

Combinatorics

Lectured by I. B. Leader, Michaelmas Term 2008

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Books (for chapter 1)

- Bollobás, *Combinatorics* (CUP, 1986) (“Bedtime reading”)
- Anderson, *Combinatorics and Finite Sets* (OUP, 1987) (“Simple and clear”)

Pre-requisites

- basic concepts of graph theory (graph, path, Hall’s Theorem)
- integers mod p
- vector spaces

Course description (from the Part III booklet)

The flavour of the course is similar to that of the Part II Graph Theory course, although we shall not rely on many of the results from that course.

We shall study collections of subsets of a finite set, with special emphasis on size, intersection and containment. There are many very natural and fundamental questions to ask about families of subsets; although many of these remain unsolved, several have been answered using a great variety of elegant techniques.

We shall cover a number of ‘classical’ extremal theorems, such as those of Erdős-Ko-Rado and Kruskal-Katona, together with more recent results concerning isoperimetric inequalities and intersecting families. The aim of the course is to give an introduction to a very active area of mathematics.

We hope to cover the following material.

Set Systems

Definitions. Antichains; Sperner’s lemma and related results. Shadows. Compression operators and the Kruskal-Katona theorem. Intersecting families; the Erdős-Ko-Rado theorem.

Isoperimetric Inequalities

Harper’s theorem and the edge-isoperimetric inequality in the cube. Inequalities in the grid. The classical isoperimetric inequality on the sphere. The ‘concentration of measure’ phenomenon. Applications.

Intersecting Families

Katona’s t -intersecting theorem. The Ahlswede-Khachatrian theorem. Restricted intersections. The Kahn-Kalai counterexample to Borsuk’s conjecture.

Desirable Previous Knowledge

The only prerequisites are the very basic concepts of graph theory.

Introductory Reading

1. Bollobás, B., *Combinatorics*, C.U.P. 1986.

Chapter 1 : Set Systems

Let X be a set. A **set system** on X (or **family of subsets** of X) is a family $\mathcal{A} \subset \mathbb{P}(X)$.

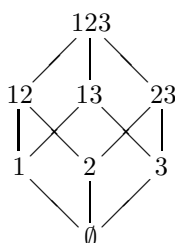
E.g., $X^{(r)} = \{A \subset X : |A| = r\}$.

Unless otherwise stated, $X = [n] = \{1, \dots, n\}$. So, e.g., $|X^{(r)}| = \binom{n}{r}$.

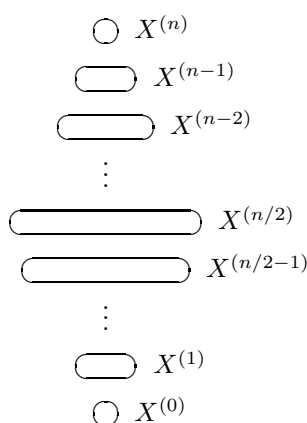
For example, $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$, where “12” = $\{1, 2\}$.

Often, we make $\mathbb{P}(X)$ into a graph by joining A to B if $|A \Delta B| = 1$. I.e., if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa. This graph is the **discrete cube** Q_n .

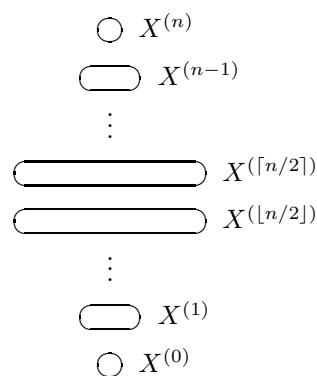
E.g., Q_3



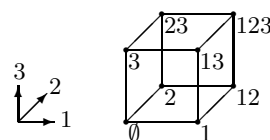
Q_n (n even)



Q_n (n odd)



If we identify a point $A \subset \mathbb{P}(X)$ with a 0–1 sequence of length n (e.g., $\{1, 3\} \mapsto 10100\dots 00$, via the indicator function $A \leftrightarrow \mathbb{I}_A$), then Q_n is precisely the unit cube in \mathbb{R}^n .



Chains and Antichains

$\mathcal{A} \subset \mathbb{P}(X)$ is a **chain** if for all $A, B \in \mathcal{A}$, either $A \subset B$ or $B \subset A$.

E.g., $\{12, 1257, 12357\}$.

$\mathcal{A} \subset \mathbb{P}(X)$ is an **antichain** if for all $A, B \in \mathcal{A}$, $A \neq B$, we have $A \not\subset B$.

E.g., $\{1, 346, 2589\}$.

How large can a chain be? We can have $|\mathcal{A}| = n + 1$, e.g. $\mathcal{A} = \{\emptyset, 1, 12, 123, \dots, [n]\}$. We cannot beat this, as a chain can meet a “level” $[X]^{(r)}$ in at most one point.

How large can an antichain be? We can have $|\mathcal{A}| = n$, e.g. $\mathcal{A} = \{1, 2, 3, \dots, n\}$. Indeed, $X^{(r)}$ is always an antichain, so we can achieve size $\binom{n}{n/2}$ (for n even) or $\binom{n}{\lfloor n/2 \rfloor}$ (for n odd).

Can we beat that?

Theorem 1 (Sperner's Lemma). Let $\mathcal{A} \subset \mathbb{P}(X)$ be an antichain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof. We'll decompose $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, and then we're done.

To do this, it's sufficient to show that:

- (i) for each $r < n/2$, there is a matching (a set of disjoint edges) from $X^{(r)}$ to $X^{(r+1)}$
- (ii) for each $r > n/2$, there is a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then just put together these matchings to form our chains. Taking complements, it's sufficient to prove (i).

Consider the subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$. This is bipartite, and has $d(A) = n - r$ for all $A \in X^{(r)}$, and $d(A) = r + 1$ for all $A \in X^{(r+1)}$.

Given $S \subset X^{(r)}$, the number of S - $\Gamma(S)$ edges = $|S|(n - r)$ (counting from below), and is $\leq |\Gamma(S)|(r + 1)$ (counting from above).

Thus, $|\Gamma(S)| \geq |S| \left(\frac{n - r}{r + 1} \right) \geq |S|$, as $r < n/2$.

So there exists a matching, by Hall's Theorem. □

Remarks. 1. We can achieve $|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}$. E.g., $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$.

2. Uniqueness? When can $|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}$? The above proof tells us nothing.

Aim. If \mathcal{A} is an antichain, then $\sum_{r=0}^n \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1$.

"Sum of percentages of levels filled is at most 1."

This trivially implies Sperner.

For $\mathcal{A} \subset X^{(r)}$, the **shadow** or **lower shadow** of \mathcal{A} is:

$$\partial\mathcal{A} = \partial^-\mathcal{A} = \{B \in X^{r-1} : B \cup \{i\} \in \mathcal{A}, \text{ some } i \in X\} \subset X^{r-1}.$$

E.g., if $\mathcal{A} = \{123, 124, 234, 135\}$, then $\partial\mathcal{A} = \{12, 13, 23, 14, 24, 34, 15, 35\}$.

Proposition 2 (Local LYM). Let $1 \leq r \leq n$, and let $\mathcal{A} \subset X^{(r)}$. Then $\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}$.

"The fraction occupied by $\partial\mathcal{A}$ is \geq that for \mathcal{A} ."

("LYM" = Lubell, Yamamoto, Meshalkin.)

Proof. The number of edges from \mathcal{A} to $\partial\mathcal{A}$ is equal to $|\mathcal{A}|r$ (counting from \mathcal{A}), and is $\leq |\partial\mathcal{A}|(n-r+1)$ (counting from $\partial\mathcal{A}$).

Thus $|\mathcal{A}|r \leq |\partial\mathcal{A}|(n-r+1)$, so $\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1}$. But $\frac{\binom{n}{r-1}}{\binom{n}{r}} = \frac{r}{n-r+1}$. \square

Equality in Local LYM? Must have that: $\forall A \in \mathcal{A}, \forall i \in A, \forall j \notin A$, we have $A - \{i\} \cup \{j\} \in \mathcal{A}$. Thus $\mathcal{A} = \emptyset$ or $X^{(r)}$.

Theorem 3 (LYM inequality). Let $\mathcal{A} \subset \mathbb{P}(X)$ be an antichain. Then $\sum_{r=0}^n \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1$.

Proof 1. “Bubble down using Local LYM.”

Write \mathcal{A}_r for $\mathcal{A} \cap X^{(r)}$. Firstly, we have $\frac{|\mathcal{A}_n|}{\binom{n}{n}} \leq 1$.

Now, $\partial\mathcal{A}_n$ and \mathcal{A}_{n-1} are disjoint subsets of X^{n-1} (as we are in an antichain).

So $\frac{|\partial\mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial\mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1$. So by Local LYM, $\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1$.

Also, $\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})$ and \mathcal{A}_{n-2} are disjoint, so $\frac{|\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1$.

Thus, by Local LYM again, $\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1$.

Keep going. We get $\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \dots + \frac{|\mathcal{A}_0|}{\binom{n}{0}} \leq 1$. \square

Equality in LYM? We must have had equality in each application of Local LYM. Hence, for the greatest r with $\mathcal{A}_r \neq \emptyset$, we must have $\mathcal{A}_r = X^{(r)}$, whence $\mathcal{A} = X^{(r)}$ as \mathcal{A} is an antichain.

Conclusion: equality in LYM $\Leftrightarrow \mathcal{A} = X^{(r)}$, some r .

In particular, equality in Sperner $\Leftrightarrow \begin{cases} \mathcal{A} = X^{(n/2)} & (n \text{ even}) \\ \mathcal{A} = X^{(\lfloor n/2 \rfloor)} \text{ or } X^{(\lceil n/2 \rceil)} & (n \text{ odd}) \end{cases}$

Proof 2. Choose, uniformly at random, a maximal chain \mathcal{C} (i.e., $A_0 \subset A_1 \subset \dots \subset A_n$, with $|A_i| = i$ for all i).

For a fixed set A , we have $P(\mathcal{C} \text{ meets } \mathcal{A}_r) = \frac{|\mathcal{A}_r|}{\binom{n}{r}}$ (as events are disjoint).

So $P(\mathcal{C} \text{ meets } \mathcal{A}) = \sum_{r=0}^n \frac{|\mathcal{A}_r|}{\binom{n}{r}}$ (as events are disjoint). Thus $\sum_{r=0}^n \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leq 1$. \square

Equivalently, the number of maximal chains = $n!$, and the number of maximal chains through a given r -set = $r!(n-r)!$. So $\sum_{r=0}^n |\mathcal{A}_r| r!(n-r)! \leq n!$.

Shadows

If $\mathcal{A} \subset X^{(r)}$, we know $|\partial\mathcal{A}| \geq |\mathcal{A}| \frac{r}{n-r+1}$, but equality is rare (only for $\mathcal{A} = \emptyset$ or $X^{(r)}$).

What happens in between?

How would we choose $\mathcal{A} \subset X^{(r)}$, with $|\mathcal{A}|$ given, to minimise $|\partial\mathcal{A}|$? It's believable that for $|\mathcal{A}| = \binom{k}{r}$ for some k , we should take $\mathcal{A} = [k]^{(r)}$, yielding $\partial\mathcal{A} = [k]^{(r-1)}$.

What if $[k]^{(r)} < \mathcal{A} < [k+1]^{(r)}$? It's believable that we would take $[k]^{(r)}$ and some extra sets from $[k+1]^{(r)}$.

E.g., for $\binom{7}{3} + \binom{4}{2}$ 3-sets, we would try $\mathcal{A} = [7]^{(3)} \cup \{A \cup \{8\} : A \in [4]^{(2)}\}$.

Two total orderings on $X^{(r)}$

We're given $A, B \in X^{(r)}$, say $A = \{a_1, \dots, a_r\}$ where $a_1 < \dots < a_r$, and $B = \{b_1, \dots, b_r\}$ where $b_1 < \dots < b_r$.

Say that $A < B$ in the **lexicographic** or **lex** order if there exists i with $a_i < b_i$ and $a_j = b_j$ for all $j < i$. Equivalently, $A < B$ is $a_i < b_i$ where $i = \min\{j : a_j \neq b_j\}$.

“Use small elements if possible.”

E.g., lex on $[4]^{(2)}$: 12, 13, 14, 23, 24, 34.

lex on $[6]^{(3)}$: 123, 124, 125, 126, 134, 135, 136, 145, 146, 156,
234, 235, 236, 245, 246, 256, 345, 346, 356, 456.

Say that $A < B$ in the **colexicographic** or **colex** order if there exists i with $a_i < b_i$ and $a_j = b_j$ for all $j > i$. Equivalently, $A < B$ is $a_i < b_i$ where $i = \max\{j : a_j \neq b_j\}$.

“Don't use large elements.”

Equivalently, $A < B$ if $\sum_{i \in A} 2^i < \sum_{i \in B} 2^i$.

E.g., colex on $[4]^{(2)}$: 12, 13, 23, 14, 24, 34.

colex on $[6]^{(3)}$: 123, 124, 134, 234, 125, 135, 235, 145, 245, 345,
126, 136, 236, 146, 246, 346, 156, 256, 356, 456.

Note that $[m]^{(r)}$ is an initial segment (i.e., first t elements, for some t) of $[m+1]^{(r)}$ in colex. Thus we could view colex as an enumeration of $\mathbb{N}^{(r)}$. (This is false for lex.)

Aim. Initial segments of colex have smallest shadow ∂ .

I.e., if $\mathcal{A} \subset X^{(r)}$ and $\mathcal{C} \subset X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$, then $|\partial\mathcal{C}| \leq |\partial\mathcal{A}|$.

In particular, $|\mathcal{A}| = \binom{k}{r} \Rightarrow |\partial\mathcal{A}| \geq \binom{k}{r-1}$.

Compressions

Given $\mathcal{A} \subset X^{(r)}$, we would like to replace \mathcal{A} by some $\mathcal{A}' \subset X^{(r)}$, where:

- (i) $|\mathcal{A}'| = |\mathcal{A}|$, (ii) $|\partial\mathcal{A}'| \leq |\partial\mathcal{A}|$, (iii) \mathcal{A}' “looks more like” \mathcal{C} than \mathcal{A} did.

We would like to find several such “compression” operations: $\mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}'' \rightarrow \dots \rightarrow \mathcal{B}$, for which either $\mathcal{B} = \mathcal{C}$, or \mathcal{B} is so similar to \mathcal{C} that we can see directly that $|\partial\mathcal{B}| \geq |\partial\mathcal{C}|$.

“Colex prefers 1 to 2” inspires the following.

Fix $1 \leq i < j \leq n$. The ij -**compression** is defined as follows.

For $A \in X^{(r)}$, let $C_{ij}(A) = \begin{cases} A \cup i - j & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}$

And for $\mathcal{A} \subset X^{(r)}$, let $C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$.

I.e., “replace j by i if possible”.

E.g., if $\mathcal{A} = \{123.124.135.235.245.367\}$ then $C_{ij}(\mathcal{A}) = \{123, 124, 135, 235, 145, 267\}$.

We clearly have $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$. Say that \mathcal{A} is ij -**compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}$.

Lemma 4. Let $\mathcal{A} \subset X^{(r)}$. Then for any $1 \leq i < j \leq n$, we have $|\partial C_{ij}(\mathcal{A})| \leq |\partial\mathcal{A}|$.

Proof. Write \mathcal{A}' for $C_{ij}(\mathcal{A})$. We’ll show that for each $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we have $j \notin B$, $i \in B$, and $B \cup j - i \in \partial\mathcal{A} - \partial\mathcal{A}'$. (Then done.)

We have $B \cup x \in \mathcal{A}'$ for some x , and $B \cup x \notin \mathcal{A}$, so $i \in B \cup x$, $j \notin B \cup x$. We cannot have $i = x$, as then $B \cup i \in \mathcal{A}'$, so either $B \cup i$ or $B \cup j$ is in \mathcal{A} , so $B \in \partial\mathcal{A}$, contradiction. Thus we know $i \in B$ and $j \notin B$. Since $B \cup x \cup j - i \in \mathcal{A}$, we certainly have $B \cup j - i \in \partial\mathcal{A}$.

Claim. $B \cup j - i \notin \partial\mathcal{A}'$.

Proof of claim. Suppose $(B \cup j - i) \cup y \in \mathcal{A}$ for some y . Then we cannot have $y = i$, or else $B \cup j \in \mathcal{A}'$, whence $B \cup j \in \mathcal{A}$ and so $B \in \partial\mathcal{A}$, contradiction.

So $i \notin (B \cup j - i) \cup y$ and $j \in (B \cup j - i) \cup y$. And so $(B \cup j - i) \cup y$ and $B \cup y$ are both in \mathcal{A} (by definition of \mathcal{A}'). So $B \in \partial\mathcal{A}$, contradiction \square

Remark. We actually showed $\partial C_{ij}(\mathcal{A}) \subset C_{ij}(\partial\mathcal{A})$.

Say that \mathcal{A} is **left-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}$ for all $i < j$.

Corollary 5. Let $\mathcal{A} \subset X^{(r)}$. Then there exists $\mathcal{B} \subset X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$, $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$, and \mathcal{B} is left-compressed.

Proof. Define a sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots \subset X^{(r)}$ as follows. Set $\mathcal{A}_0 = \mathcal{A}$. Having chosen $\mathcal{A}_0, \dots, \mathcal{A}_k$, if \mathcal{A}_k is left-compressed, stop the sequence with \mathcal{A}_k . If not, choose $i < j$ with \mathcal{A}_k not ij -compressed, and set $\mathcal{A}_{k+1} = C_{ij}(\mathcal{A}_k)$. This must terminate – e.g., as the function $\sum_{A \in \mathcal{A}_k} \sum_{x \in A} x$ is decreasing in k . The system $\mathcal{B} = \mathcal{A}_k$ has $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$ by Lemma 4. \square

Remarks. 1. Alternatively, among all $\mathcal{B} \subset X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$, choose one with minimal $\sum_{A \in \mathcal{B}} \sum_{x \in A} x$.

2. It is possible to apply each C_{ij} at most once if we choose the order sensibly.

Any initial segment of colex is left-compressed. However, the converse is false, for example $\mathcal{A} = \{123, 124, 125, 126, 127\}$.

“Colex prefers 23 to 14” inspires the following.

For $U, V \subset X$ with $|U| = |V|$ and $U \cap V = \emptyset$, define the UV -**compression** as follows.

For $A \in X^{(r)}$, let $C_{UV}(A) = \begin{cases} A \cup U - V & \text{if } V \subset A, A \cap U = \emptyset \\ A & \text{otherwise} \end{cases}$

And for $\mathcal{A} \subset X^{(r)}$, let $C_{UV}(\mathcal{A}) = \{C_{UV}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{UV}(A) \in \mathcal{A}\}$.

E.g., if $\mathcal{A} = \{123, 145, 147, 234, 235, 267\}$ then $C_{23,14}(\mathcal{A}) = \{123, 145, 237, 234, 235, 267\}$.

Note that $|C_{UV}(\mathcal{A})| = |\mathcal{A}|$, and that $C_{\{i\},\{j\}}(\mathcal{A}) = C_{ij}(\mathcal{A})$.

Unfortunately, we can have $|\partial C_{UV}(\mathcal{A})| > |\partial\mathcal{A}|$. E.g., let $\mathcal{A} = \{147, 478\}$. Then $|\partial\mathcal{A}| = 5$, but $C_{23,14}(\mathcal{A}) = \{237, 478\}$ with $|C_{23,14}(\mathcal{A})| = 6$.

Say \mathcal{A} is UV -**compressed** is $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Lemma 6. Let $U, V \subset X$ be disjoint, with $|U| = |V|$. Let $\mathcal{A} \subset X^{(r)}$. Suppose that for all $u \in U$, there is $v \in V$ such that \mathcal{A} is $(U-u, V-v)$ -compressed. Then $|\partial C_{UV}(\mathcal{A})| \leq |\partial\mathcal{A}|$.

Proof. Write \mathcal{A}' for $C_{UV}(\mathcal{A})$. For any $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we'll show that $U \subset B$, $V \cap B = \emptyset$, and $B \cup V - U \in \partial\mathcal{A} - \partial\mathcal{A}'$. (Then done.)

We have $B \cup x \in \mathcal{A}'$ for some x , and $B \cup x \notin \mathcal{A}$. Thus $U \subset B \cup x$, $V \cap (B \cup x) = \emptyset$, and $B \cup x \cup V - U \in \mathcal{A}$. So certainly $V \cap B = \emptyset$.

Also, $x \notin U$. For if $x \in U$, then there is $y \in V$ with \mathcal{A} being $(U-x, V-y)$ -compressed. But $B \cup x \cup V - U \in \mathcal{A}$, so then $B \cup y \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. Thus $U \subset B$.

And we have $B \cup V - U \in \partial\mathcal{A}$, because $B \cup x \cup V - U \in \mathcal{A}$.

Suppose $B \cup V - U \in \partial\mathcal{A}'$. So $(B \cup V - U) \cup w \in \mathcal{A}'$ for some w .

If $w \notin U$, then because $(B \cup V - U) \cup w \in \mathcal{A}'$, we must have $(B \cup V - U) \cup w$ and $B \cup w$ in \mathcal{A} (by definition of C_{UV}), contradicting $B \notin \partial\mathcal{A}$.

If $w \in U$, then we have that \mathcal{A} is $(U-w, V-z)$ -compressed for some $z \in V$. So from $(B \cup V - U) \cup w \in \mathcal{A}$ (which is true as it contains V , so cannot have moved), we obtain $B \cup z \in \mathcal{A}$, a contradiction. \square

Remark. We actually showed that $\partial C_{UV}(\mathcal{A}) \subset C_{UV}(\partial\mathcal{A})$.

Theorem 7 (Kruskal-Katona Theorem). Let $\mathcal{A} \subset X^{(r)}$ ($1 \leq r \leq n$), and let \mathcal{C} be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial\mathcal{A}| \geq |\partial\mathcal{C}|$.

In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial\mathcal{A}| \leq \binom{k}{r-1}$.

Proof. Let $\Gamma = \{(U, V) : U, V \subset X, |U| = |V| > 0, U \cap V = \emptyset, \max V > \max U\}$.

Define a sequence $\mathcal{A}_0, \mathcal{A}_1, \dots$ as follows. Set $\mathcal{A}_0 = \mathcal{A}$. Having chosen $\mathcal{A}_0, \dots, \mathcal{A}_k$, if \mathcal{A}_k is UV -compressed, stop the sequence with \mathcal{A}_k . If not, choose $(U, V) \in \Gamma$, with $|U|$ minimal, such that \mathcal{A}_k is not UV -compressed, and set $\mathcal{A}_{k+1} = C_{UV}(\mathcal{A}_k)$.

For each $u \in U$, setting $v = \min V$, we have $(U - u, V - v) \in \Gamma \cup \{0, 0\}$. So \mathcal{A}_k is $(U - u, V - v)$ -compressed. Thus by Lemma 6, we have $|\partial\mathcal{A}_{k+1}| \leq |\partial\mathcal{A}_k|$. Continue.

This sequence must terminate, as $\sum_{A \in \mathcal{A}_k} \sum_{i \in A} 2^i$ is decreasing. The final term, $\mathcal{B} = \mathcal{A}_k$, satisfies: $|\mathcal{B}| = |\mathcal{A}|$, $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$, and \mathcal{B} is UV -compressed for all $(U, V) \in \Gamma$.

Claim. $\mathcal{B} = \mathcal{C}$.

Proof of claim. Suppose not. Then there exists $A, B \in X^{(r)}$ with $A < B$ in colex, and $A \notin \mathcal{B}$, $B \in \mathcal{B}$. Set $U = A - B$, $V = B - A$. We have $\max V > \max U$ (as $A < B$ in colex). Thus $(U, V) \in \Gamma$. So $B \in \mathcal{B} \Rightarrow A \in \mathcal{B}$. Contradiction. \square

Remarks. 1. Equivalently, if $|\mathcal{A}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \dots + \binom{k_s}{s}$, where $k_r > k_{r-1} > \dots > k_s$ and $s > 0$, then $|\partial\mathcal{A}| \geq \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \dots + \binom{k_s}{s-1}$.

2. The proof used only Lemma 6, and not Lemma 4 or Corollary 5.

3. Equality? We can check that if $|\mathcal{A}| = \binom{k}{r}$ and we have equality in Kruskal-Katona (i.e., $|\partial\mathcal{A}| = \binom{k}{r-1}$), then $\mathcal{A} = Y^{(r)}$ for some $Y \subset X$ with $|Y| = k$. But it is false in general that if equality holds (i.e., $|\partial\mathcal{A}| = |\partial\mathcal{C}|$) then \mathcal{A} is isomorphic to \mathcal{C} .

(\mathcal{A}, \mathcal{B} are **isomorphic** if there is a permutation $f : X \rightarrow X$ sending \mathcal{A} to \mathcal{B} .)

For $\mathcal{A} \subset X^{(r)}$ ($0 \leq r \leq n-1$), the **upper shadow** of \mathcal{A} is $\partial^+\mathcal{A} \subset X^{(r+1)}$, given by $\partial^+\mathcal{A} = \{A \cup x : A \in \mathcal{A}, x \in X, x \notin A\}$.

Now, $A < B$ in colex on $X^{(r)} \Leftrightarrow A^c < B^c$ in lex on $X^{(n-r)}$ with the ground set order reversed.

Corollary 8. Let $\mathcal{A} \subset X^{(r)}$ ($0 \leq r \leq n-1$) and let \mathcal{C} be the initial segment of lex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^+\mathcal{C}| \leq |\partial^+\mathcal{A}|$.

Proof. Take complements. \square

The shadow of an initial segment of colex on $X^{(r)}$ is an initial segment of colex on $X^{(r-1)}$ (for if $\mathcal{C} = \{A \in X^{(r)} : A \leq a_1 \dots a_r\}$ then $\partial\mathcal{C} = \{B \in X^{(r-1)} : B \leq a_2 \dots a_r\}$), so we have:

Corollary 9. Let $\mathcal{A} \subset X^{(r)}$, and let \mathcal{C} be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^t\mathcal{A}| \geq |\partial^t\mathcal{C}|$ for all $1 \leq t \leq r$. In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial^t\mathcal{A}| \geq \binom{k}{r-t}$.

Proof. If $|\partial^t\mathcal{A}| \geq |\partial^t\mathcal{C}|$ then $|\partial^{t+1}\mathcal{A}| \geq |\partial^{t+1}\mathcal{C}|$ by Kruskal-Katona. \square

Intersecting Families

Say that $\mathcal{A} \subset \mathbb{P}(X)$ is **intersecting** if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$.

How large can an intersecting family be?

We can have $|\mathcal{A}| = 2^{n-1}$, e.g. $\mathcal{A} = \{A \subset X : 1 \in A\}$.

Proposition 10. Let $\mathcal{A} \subset \mathbb{P}(X)$ be intersecting. Then $|\mathcal{A}| \leq 2^{n-1}$.

Proof. For any $A \in \mathbb{P}(X)$, at most one of A, A^c can belong to \mathcal{A} . □

Remark. There are many extremal systems. E.g., $\{A \in \mathbb{P}(X) : |A| > n/2\}$ for n odd.

What if we restrict to $\mathcal{A} \subset X^{(r)}$?

If $r > n/2$, silly: we can take $\mathcal{A} = X^{(r)}$.

If $r = n/2$, take one of each complementary pair A, A^c – gives $\frac{1}{2} \binom{n}{r}$. (Optimal, as we can never take both of A and A^c .)

So assume $r < n/2$.

Obvious guess: $\mathcal{A} = \{A \in X^{(r)} : 1 \in A\}$. This has $|\mathcal{A}| = \binom{n-1}{r-1} = \frac{r}{n} \binom{n}{r}$.

We could also try, e.g., $\mathcal{B} = \{A \in X^{(r)} : |A \cap \{1, 2, 3\}| \geq 2\}$.

E.g., in $[8]^{(3)}$, $|\mathcal{A}| = 21$ and $|\mathcal{B}| = 1 + 3 \cdot 5 = 16 < 21$.

Theorem 11 (Erdős-Ko-Rado Theorem). Let $r < n/2$, and $\mathcal{A} \subset X^{(r)}$ be intersecting.

Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.

Proof. “Bubble down with Kruskal-Katona.”

For any $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$. I.e., $A \not\subset B^c$. Thus, writing $\overline{\mathcal{A}} = \{B^c : B \in \mathcal{A}\} \subset X^{(n-r)}$, we have that \mathcal{A} and $\partial^{n-2r} \overline{\mathcal{A}}$ are *disjoint* subsets of $X^{(r)}$.

Suppose that $|\mathcal{A}| > \binom{n-1}{r-1}$.

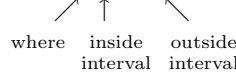
Then $|\overline{\mathcal{A}}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$. So $|\partial^{n-2r} \overline{\mathcal{A}}| \geq \binom{n-1}{r}$, by Corollary 9.

Thus $|\mathcal{A}| + |\partial^{n-2r} \overline{\mathcal{A}}| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$. Contradiction. □

Note. Numbers *had* to work out, given that if $\mathcal{A} = \{A \in X^{(r)} : 1 \in A\}$, we have equality, and $\mathcal{A}, \partial^{n-2r} \overline{\mathcal{A}}$ partition $X^{(r)}$.

Proof 2. Consider a cyclic ordering of X , i.e. a bijection $c : \mathbb{Z} \rightarrow X$. How many $A \in \mathcal{A}$ are *intervals* (blocks of r consecutive elements) in this ordering? Answer: at most r . Indeed, suppose $\{c_1, c_2, \dots, c_r\} \in \mathcal{A}$. Then for each $1 \leq i \leq r-1$, at most one of $\{c_{i-r+1}, \dots, c_i\}$ and $\{c_{i+1}, \dots, c_{i+r}\}$ belongs to \mathcal{A} .

Also, each r -set is an interval in precisely $n r! (n-r)!$ of the $n!$ cyclic orderings.



Thus, $|\mathcal{A}| n r! (n-r)! \leq n! r$, and so $|\mathcal{A}| \leq \frac{r}{n} \binom{n}{r}$. □.

Notes. 1. Equivalently, we are double-counting the edges in the bipartite graph with vertex classes \mathcal{A} and the cyclic orderings, where we join $A \in \mathcal{A}$ to a cyclic ordering c if A is an interval in c .

2. This method is called **averaging**, or **Katona's method**.

Equality in Erdős-Ko-Rado

Want: if $\mathcal{A} \in X^{(r)}$ is intersecting ($r < n/2$) and $|\mathcal{A}| = \binom{n-1}{r-1}$, then $\mathcal{A} = \{A \in X^{(r)} : i \in A\}$, some i .

From proof 2, for each cyclic ordering c , we have r intervals in \mathcal{A} . These must be all of the r intervals containing a point $x(c)$, say. Our task is to show that $x(c) = x(c')$ for all c, c' .

Fix a cyclic ordering c , and say $x(c) = c_0$.

Thus $\{c_0, \dots, c_{r-1}\}, \{c_{-r+1}, \dots, c_0\} \in \mathcal{A}$, and $\{c_1, \dots, c_r\}, \{c_{-r}, \dots, c_{-1}\} \notin \mathcal{A}$.

Let c' be obtained from c by swapping two adjacent elements $\neq c_0$. Say we swap c_i and c_{i+1} . Wlog (else reflect) $i \geq (n-1)/2$. (Not $n/2$, in case $i = (n-1)/2, i+1 = (n+1)/2$.)

Then $\{c_0, \dots, c_{r-1}\}$ is an interval of c' , and $\{c_1, \dots, c_r\}$ is an interval of c' (unless $r = (n-1)/2$ and $i = (n-1)/2$, in which case $\{c_1, \dots, c_{r-1}, c_{r+1}\}$ is an interval of c').

But $\{c_0, \dots, c_{r-1}\} \in \mathcal{A}$ and $\{c_1, \dots, c_r\} \notin \mathcal{A}$ (and $\{c_1, \dots, c_{r-1}, c_{r+1}\} \notin \mathcal{A}$, as disjoint from $\{c_{-r+1}, \dots, c_0\} \in \mathcal{A}$).

Thus $x(c') = c_0$. Hence $x(c) = x(c')$ for all c, c' , as required.

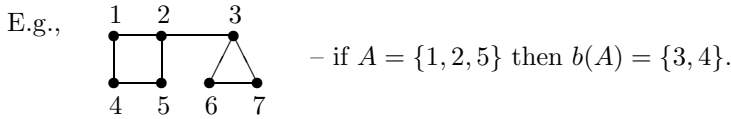
Chapter 2 : Isoperimetric Inequalities

“How small can the boundary of a set of given size be?”

- E.g., – among subsets of \mathbb{R}^2 of given area, the disc has smallest perimeter.
- among subsets of \mathbb{R}^3 of given volume, solid sphere has smallest surface area.
- among subsets of S^2 of given area, circular cap has smallest perimeter.

For a graph G and $A \subset V(G)$, the **boundary** of A is:

$$b(A) = \{x \in G : x \notin A, xy \in E(G) \text{ for some } y \in A\}.$$

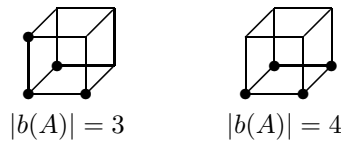


An **isoperimetric inequality** on G is an inequality of the form: $|b(A)| \geq |f(A)|$ for all $A \subset V(G)$.

Equivalently, minimise the **neighbourhood** of A , $N(A) = A \cup b(A) = \{x : d(x, y) \leq 1\}$, where d = usual graph distance. Often a good guess is $B(x, r) = \{y : d(x, y) \leq r\}$

What happens in Q_n ?

E.g., $|A| = 4$ in Q_3 .



Guess: $B(\emptyset, r) = X^{(\leq r)} = X^{(0)} \cup \dots \cup X^{(r)}$ are best.

What if $|X^{(\leq r)}| < |A| < |X^{(\leq r+1)}|$? Guess: take $A = X^{(\leq r)} \cup B$, some $B \subset X^{(r+1)}$. Then $b(A) = (X^{(r+1)} - B) \cup \partial^+ B$, so we'd take B an initial segment of lex, by Kruskal-Katona.

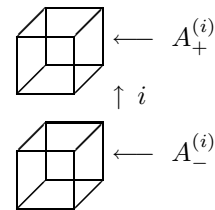
This suggests the following. The **simplicial ordering** on $\mathbb{P}(X)$ is defined by: $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x < y$ in lex.

Aim. Initial segments of the simplicial order are best.

Let $A \subset Q_n$, and let $1 \leq i \leq n$.

The **i -sections** of A are the sets $A_+^{(i)}, A_-^{(i)} \subset \mathbb{P}(X - i)$ given by

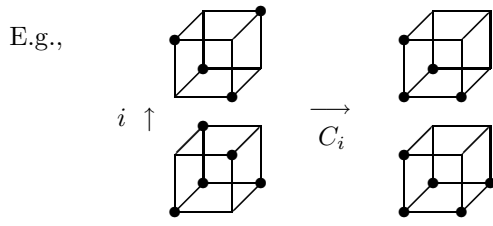
$$A_-^{(i)} = \{x \in A : i \notin x\}, \quad A_+^{(i)} = \{x - i : x \in A, i \in x\}.$$



The **i -compression** $C_i(A)$ of A is defined by giving its i -sections:

$(C_i(A))_+^{(i)}$ is the initial segment of the simplicial order on $\mathbb{P}(X - i)$ of size $A_+^{(i)}$.

$(C_i(A))_-^{(i)}$ is the initial segment of the simplicial order on $\mathbb{P}(X - i)$ of size $A_-^{(i)}$.



Note that $|C_i(A)| = |A|$, and also that $C_i(A)$ “looks more like” a Hamming ball than A did. (A **Hamming ball** is a set $A \subset Q_n$ with $X^{(\leq r)} < A < X^{(\leq r+1)}$, some r .)

Say A is i -**compressed** if $C_i(A) = A$.

Theorem 1 (Harper’s Theorem). Let $A \subset Q_n$ and let C be the initial segment of the simplicial order with $|C| = |A|$. Then $|N(A)| \geq |N(C)|$.

In particular, if $|A| \geq \sum_{i=0}^r \binom{n}{i}$ then $|N(A)| \geq \sum_{i=0}^{r+1} \binom{n}{i}$.

Remarks. 1. If we knew that A was a Hamming ball, we would be done by Kruskal-Katona.

2. Conversely, Theorem 1 implies Kruskal-Katona: given some $B \subset X^{(r)}$, apply Theorem 1 to $A = X^{(\leq r)} \cup B$.

Proof. Induction on n . Done if $n = 1$. Given $A \subset Q_n$ ($n > 1$), $1 \leq i \leq n$, claim:

Claim. $|N(C_i(A))| \leq |N(A)|$.

Proof of claim. Write B for $C_i(A)$. We have $|N(A)| = |N(A_-) \cup A_+| + |N(A_+) \cup A_-|$ and $|N(B)| = |N(B_-) \cup B_+| + |N(B_+) \cup B_-|$.

Now, $|B_+| = |A_+|$, and $|N(B_-)| \leq |N(A_-)|$, by induction. But B_+ is an initial segment of the simplicial order, and so is $N(B_-)$ (as the neighbourhood of an initial segment is an initial segment). Thus B_+ and $N(B_-)$ are nested (i.e., one is contained in the other).

And so certainly $|N(B_-) \cup B_+| \leq |N(A_-) \cup A_+|$.

Similarly, we have $|N(B_+) \cup B_-| \leq |N(A_+) \cup A_-|$. Thus $|N(B)| \leq |N(A)|$.

Define A_0, A_1, \dots as follows. Set $A_0 = A$. Having chosen A_0, \dots, A_k , if A_k is i -compressed for all i then stop the sequence with A_k .

If not, choose i with $C_i(A_k) \neq A_k$, set $A_{k+1} = C_i(A_k)$ and continue. This must terminate, as $\sum_{x \in A_k} f(x)$ is decreasing, where $f(x)$ denotes the position of x in the simplicial order.

Then $B = A_k$ satisfies: $|B| = |A|$, $|N(B)| \leq |N(A)|$, and B is i -compressed for all i .

Does B being i -compressed for all i imply that B is an initial segment of the simplicial order? (If so then we are done, and $B = C$.)

Answer: no, e.g. $\{\emptyset, 1, 2, 12\} \subset Q_n$. However, we are done by Lemma 2 below.

Lemma 2. Let $B \subset Q_n$ be i -compressed for all i , but not an initial segment of the simplicial order. Then for n odd, say $n = 2k + 1$, we have

$$B = X^{(\leq k)} - \{k + 2, k + 2, \dots, 2k + 1\} \cup \{1, 2, 3, \dots, k + 1\},$$

and for n even, say $n = 2k$, we have

$$B = X^{(< k)} \cup \{x \in X^{(k)} : 1 \in x\} - \{1, k + 2, k + 3, \dots, 2k\} \cup \{2, 3, \dots, k + 1\}.$$

(Then done, as in each case we have $|N(B)| \geq |N(C)|$.)

Proof. We have some $x < y$ with $x \notin B$, $y \in B$. For each i , we cannot have $i \in x$ and $i \in y$, as B is i -compressed. Similarly, we cannot have $i \notin x$ and $i \notin y$. Thus $x = y^c$.

So for each $x \notin B$, we have at most one later point $y \in B$ (namely x^c), and for each $y \in B$, we have at most one earlier point $x \notin B$ (namely y^c).

So $B = \{z : z \leq y\} - \{x\}$, where x is the predecessor of y , and $x = y^c$.

If n is odd, we must have x the last $\frac{n-1}{2}$ set.

If n is even, we must have x the last $\frac{n}{2}$ set containing 1. So done. \square

Notes. 1. Can also prove Harper's Theorem by UV -compressions.

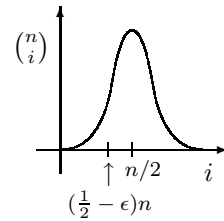
2. Can also use these "codimension-1" compressions to prove Kruskal-Katona.

For $A \subset Q_n$, the t -neighbourhood of A is $N^t(A) = A_{(t)} = \{x \in Q_n : d(x, A) \leq t\}$.

Corollary 3. Let $A \subset Q_n$ with $|A| \geq \sum_{i=0}^r \binom{n}{i}$. Then for $1 \leq t \leq n - r$, $|N^t(A)| \geq \sum_{i=0}^{r+t} \binom{n}{i}$.

Proof. Theorem 1, plus induction. \square

To get a feel for the strength of Corollary 3, we'll need some estimates on $\sum_{i=0}^r \binom{n}{i}$, etc.



Proposition 4. Let $0 < \epsilon < 1/4$. Then $\sum_{i=0}^{\lfloor (\frac{1}{2}-\epsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{\epsilon} e^{-\epsilon^2 n/2} \cdot 2^n$

Note. This is an *exponentially small* fraction of 2^n (for ϵ fixed, $n \rightarrow \infty$).

(“This is $\sim \epsilon\sqrt{n}$ standard deviations from the mean $n/2$.”)

Proof. $\binom{n}{i-1} = \binom{n}{i} \frac{i}{n+i-1}$. So for $i \leq \lfloor \frac{1}{2}(1-\epsilon)n \rfloor$, we have

$$\frac{\binom{n}{i-1}}{\binom{n}{i}} = \frac{i}{n-i+1} \leq \frac{(\frac{1}{2}-\epsilon)n}{(\frac{1}{2}+\epsilon)n} = \frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon} = 1 - \frac{2\epsilon}{\frac{1}{2}+\epsilon} \leq 1 - 2\epsilon.$$

So, $\sum_{i=0}^{\lfloor (\frac{1}{2}-\epsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\epsilon} \binom{n}{\lfloor (\frac{1}{2}-\epsilon)n \rfloor}$ (sum of a GP).

Similarly, $\binom{n}{\lfloor (\frac{1}{2}-\epsilon)n \rfloor} \leq \binom{n}{\lfloor (\frac{1}{2}-\frac{\epsilon}{2})n \rfloor} \left(1 - 2\frac{\epsilon}{2}\right)^{\epsilon n/2-1}$ (same argument, $\epsilon \rightarrow \epsilon/2$)

which is $\leq 2^n 2(1-\epsilon)^{\epsilon n/2} \leq 2^n 2e^{-\epsilon^2 n/2}$, as $(1-\epsilon) \leq e^{-\epsilon}$.

Thus, $\sum_{i=0}^{\lfloor (\frac{1}{2}-\epsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\epsilon} 2e^{-\epsilon^2 n/2} 2^n$. □

Theorem 5. Let $A \subset Q_n$, $0 < \epsilon < 1/4$. Then $\frac{|A|}{2^n} \geq \frac{1}{2} \Rightarrow \frac{|A_{(\epsilon n)}|}{2^n} \geq 1 - \frac{1}{\epsilon} e^{-\epsilon^2 n/2}$.

“ $\frac{1}{2}$ -sized sets have exponentially large neighbourhoods.”

Proof. We have $|A| \geq \sum_{i=0}^{\lceil n/2-1 \rceil} \binom{n}{i}$, so by Harper we have $|A_{(\epsilon n)}| \geq \sum_{i=0}^{\lceil n/2+\epsilon n-1 \rceil} \binom{n}{i}$.

So $|A_{(\epsilon n)}^c| \leq \sum_{i=\lceil n/2+\epsilon n \rceil}^n \binom{n}{i} = \sum_{i=0}^{\lfloor n/2-\epsilon n \rfloor} \binom{n}{i} \leq \frac{1}{\epsilon} e^{-\epsilon^2 n/2} 2^n$. □

Remark. The above is concerned with $\frac{1}{2}$ -sized sets, but the same argument would show that

$$\frac{|A|}{2^n} \geq \frac{1}{\epsilon} e^{-\epsilon^2 n/2} \Rightarrow \frac{|A_{(2\epsilon n)}|}{2^n} \geq 1 - \frac{1}{\epsilon} e^{-\epsilon^2 n/2}.$$

Concentration of Measure

Say $f : Q_n \rightarrow \mathbb{R}$ is **Lipschitz** if $|f(x) - f(y)| \leq 1$ for all x, y adjacent.

A real number M is a **median** or **Lévy mean** for f is $|\{x : f(x) \leq M\}| \geq 2^{n-1}$ and $|\{x : f(x) \geq M\}| \geq 2^{n-1}$.

We are now ready to show that “every well-behaved function on Q_n is roughly constant nearly everywhere”.

Theorem 6. Let f be a Lipschitz function on Q_n , with median M .

Then $\frac{|\{x : |f(x) - M| \leq \epsilon n\}|}{2^n} \geq 1 - \frac{2}{\epsilon} e^{-\epsilon^2 n/2}$ ($0 < \epsilon < 1/4$)

Note. This is the “concentration of measure” phenomenon.

Proof. Let $A = \{x : f(x) \leq M\}$. Then $\frac{|A|}{2^n} \geq \frac{1}{2}$, so $\frac{|A_{(\epsilon n)}|}{2^n} \geq 1 - \frac{1}{\epsilon} e^{-\epsilon^2 n/2}$.

But $x \in A_{(\epsilon n)} \Rightarrow f(x) \leq M + \epsilon n$ (as f Lipschitz).

And so $\frac{|\{x : f(x) \leq M + \epsilon n\}|}{2^n} \geq 1 - \frac{1}{\epsilon} e^{-\epsilon^2 n/2}$.

Similarly, $\frac{|\{x : f(x) \geq M - \epsilon n\}|}{2^n} \geq 1 - \frac{1}{\epsilon} e^{-\epsilon^2 n/2}$. □

Let G be a graph of diameter D . (I.e., $D = \max\{d(x, y) : x, y \in G\}$.)

Define $\alpha(G, \epsilon) = \max \left\{ 1 - \frac{|A_{(\epsilon n)}|}{|G|} : A \subset G, \frac{|A|}{|G|} \geq \frac{1}{2} \right\}$.

So “ $\alpha(G, \epsilon)$ small” says: $\frac{1}{2}$ -sized sets have big ϵD -neighbourhoods.

A sequence G_1, G_2, \dots of graphs is a **Lévy family** if $\alpha(G_n, \epsilon) \rightarrow 0$ as $n \rightarrow \infty$, for each $\epsilon > 0$.

So, e.g., Theorem 5 tells us that $(Q_n)_{n=1}^\infty$ is a Lévy family, and even a **normal** Lévy family (meaning $\alpha(G_n, \epsilon)$ is exponentially small, for any ϵ).

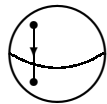
So we again have concentration of measure (“Lipschitz functions on G_n are almost constant nearly everywhere”) for any Lévy family. It turns out that many natural families of graphs are Lévy families. For example, the permutation groups S_n (made into a graph by: σ adjacent to τ if $\sigma^{-1}\tau$ is a transposition) form a Lévy family.

Similarly, we can define $\alpha(S, \epsilon)$ for any metric measure space S (of finite diameter and finite measure), so we can again define Lévy families. It turns out that many natural families of metric spaces form Lévy families.

Example. The sphere S^n .

Two ingredients.

1. Isoperimetric inequality in S^n : $|A| = |C| \Rightarrow |A_{(\epsilon)}| \geq |C_{(\epsilon)}|$ where C is a circular cap. To prove this, we can use compressions, e.g. analogue of ij -compressions.



“Stamp on your set” – i.e., replace by the bottom point, if possible.
(2-point symmetrisation)

2. Estimate: $\frac{1}{2}$ -sized circular cap has angle $\pi/2$, so its ϵ -neighbourhood is a circular cap of angle $\pi/2 + \epsilon$. But $\int_\epsilon^1 \cos^n t dt \rightarrow 0$ as $n \rightarrow \infty$, for any fixed ϵ .

We deduced concentration of measure from isoperimetric estimates. Conversely:

Proposition 7. G a graph such that for every Lipschitz $f : G \rightarrow \mathbb{R}$ of median M , we have

$$\frac{|\{x \in G : |f(x) - M| > t\}|}{|G|} \leq \alpha \quad (\text{some fixed } t, \alpha).$$

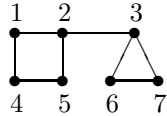
$$\text{Then } \frac{|A|}{|G|} \geq \frac{1}{2} \Rightarrow \frac{|A_{(t)}|}{|G|} \geq 1 - \alpha.$$

Proof. Let $f(x) = d(x, A)$. Then f Lipschitz, and has 0 as a median (as $|A| \geq \frac{1}{2}|G|$). \square

Edge-Isoperimetric Inequalities

For a graph G , $A \subset V(G)$, the **edge-boundary** of A is

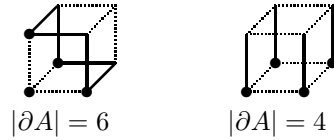
$$\partial_e A = \partial A = \{xy \in E(G) : x \in A, y \notin A\}.$$

E.g.,  - if $A = \{1, 2, 5\}$ then $\partial A = \{14, 23, 45\}$.

An **edge-isoperimetric inequality** on G in an inequality of the form $A \subset G$, $|A| = m \Rightarrow |\partial A| \geq |f(m)|$.

In the cube

E.g., $|A| = 4$ in Q_3 .



This suggests that subcubes are best.

The **binary** ordering on Q_n is given by: $x < y$ if $\max(x \Delta y) \in y$.

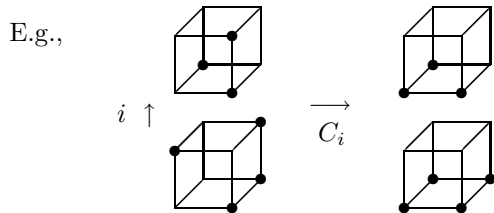
Equivalently, if $\sum_{i \in x} 2^i < \sum_{i \in y} 2^i$. "Go up in subcubes."

Aim. Initial segments of binary minimise ∂ .

For $A \subset Q_n$ and $1 \leq i \leq n$, the **i -binary-compression** $B_i(A)$ is defined by giving its i -sections:

$(B_i(A))_+^{(i)}$ is the initial segment of binary on $\mathbb{P}(X - i)$ of size $A_+^{(i)}$.

$(B_i(A))_-^{(i)}$ is the initial segment of binary on $\mathbb{P}(X - i)$ of size $A_-^{(i)}$.



Clearly $|B_i(A)| = |A|$. Say A is i -**binary-compressed** if $B_i(A) = A$.

Theorem 8 (Edge-isoperimetric inequality in the cube). Let $A \subset Q_n$, and let C be the initial segment of binary with $|C| = |A|$. Then $|\partial C| \geq |\partial A|$. In particular, $|A| = 2^k \Rightarrow |\partial A| \geq (n - k)2^k$.

Remark. Sometimes called “the theorem of Harper, Lindsey, Bernstein and Hart”.

Proof. Induction on n . Done if $n = 1$. Given $A \subset Q_n$, $1 \leq i \leq n$.

Claim. $|\partial B_i(A)| \leq |\partial A|$.

Proof of claim. Write B for $B_i(A)$.

$$\begin{array}{c} \text{Have } |\partial A| = |\partial(A_-)| + |\partial(A_+)| + |A_+ \Delta A_-|. \\ \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\ \quad \quad \text{edges in} \quad \text{edges in} \quad \text{edges} \\ \quad \quad \text{the bottom} \quad \text{the top} \quad \text{across} \end{array}$$

$$\text{Similarly, } |\partial B| = |\partial(B_-)| + |\partial(B_+)| + |B_+ \Delta B_-|.$$

Now, $|\partial(B_-)| \leq |\partial(A_-)|$ and $|\partial(B_+)| \leq |\partial(A_+)|$, by induction.

Also, $|B_+ \Delta B_-| \leq |A_+ \Delta A_-|$, because $|B_+| = |A_+|$, $|B_-| = |A_-|$, and the sets B_+ , B_- are nested, as each is an initial segment of the binary order. Thus $|\partial B| \leq |\partial A|$.

Define A_0, A_1, \dots as follows. Set $A_0 = A$. Having chosen A_0, \dots, A_k , if A_k is i -binary-compressed for i , then stop. If not, choose i with $B_i(A_k) \neq A_k$ and set $A_{k+1} = B_i(A_k)$. This must terminate, e.g. because $\sum_{x \in A_k} (\text{position of } x \text{ in binary})$ is decreasing.

The final set $B = A_k$ satisfies: $|B| = |A|$, $|\partial B| \leq |\partial A|$, and B is i -binary-compressed for all i . As before, B need not be an initial segment, e.g. $\{\emptyset, 1, 2, 12\} \subset Q_n$. But we are done by Lemma 9 below.

Lemma 9. Let $B \subset Q_n$ be i -binary-compressed for all i , but not an initial segment of binary. Then $B = \mathbb{P}(X - i) \cup \{n\} - \{1, 2, 3, \dots, n - 1\}$.

(Then done, as certainly $|\partial B| \geq |\partial C|$ in this case.)

Proof. Have some $x < y$ with $x \notin B$, $y \in B$. Then for each i , cannot have $i \in x$, $i \in y$, or $i \notin x$, $i \notin y$, as B is i -binary-compressed. So $x = y^c$.

So for each $x \notin B$, at most one $y > x$ has $y \in B$ (namely x^c), and for each $y \in B$, at most one $x < y$ has $x \notin B$ (namely y^c).

Thus $B = \{z : z \leq y\} - \{x\}$, where x is the predecessor of y , and $x = y^c$.

Hence $y = \{n\}$, as required. (“As n changes hands only once.”) □

Remark. Vital, in the above, proof, that the extremal sets in dimension $n - 1$ were *nested* (i.e., given by initial segments of some ordering).

For a graph G , the **isoperimetric number** of G is: $i(G) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset G, \frac{|A|}{|G|} \leq \frac{1}{2} \right\}$.

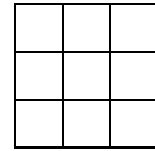
“How small can the average out-degree be?”

Corollary 10. $i(Q_n) = 1$.

Proof. The set $A = \mathbb{P}(n - 1)$ show $i(Q_n) \leq 1$. To show $i(Q_n) \geq 1$, let C be any initial segment of binary with $|C| \leq 2^{n-1}$. Then $C \subset \mathbb{P}(n - 1)$, so certainly $|\partial C| \geq |C|$. □

Inequalities in the Grid

The **grid** is the graph on $[k]^n = \{(x_1, \dots, x_n) : x_i \in \{1, \dots, k\} \forall i\}$, in which $x = (x_1, \dots, x_n)$ is joined to $y = (y_1, \dots, y_n)$ if for some i we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. (“ l_1 -norm”)



$[4]^2$

Note. For $k = 2$, this is exactly the graph Q_n .

Do Theorem 1 and Theorem 8 extend to the grid?

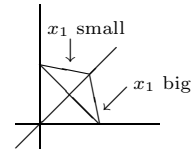
Vertex-isoperimetric inequality in the grid



Good guess: sets of the form $\{x : |x| \leq r\}$ are best, where $|x| = x_1 + \dots + x_r$.

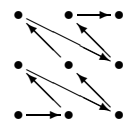
What if $|\{x : |x| \leq r\}| < |A| < |\{x : |x| \leq r + 1\}|$? We’d take A of the form $\{x : |x| \leq r\} \cup B$, some $B \subset \{x : |x| = r + 1\}$.

Define the **simplicial ordering** on $[k]^n$ by: $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x_i > y_i$ where $i = \min\{j : x_j \neq y_j\}$.



E.g., on $[3]^2$: $(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), (3, 2), (2, 3), (3, 3)$.

on $[4]^3$: $(1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2), (3, 1, 1), (2, 2, 1), (2, 1, 2), (1, 3, 1), (1, 2, 2), (1, 1, 3), (4, 1, 1), \dots$



Note. This agrees with the previous definition of simplicial for $k = 2$.

Aim. Initial segments of simplicial are best for vertex-isoperimetric.

Let $A \subset [k]^n$. For $1 \leq i \leq n$, the i -**sections** of A are the sets A_1, \dots, A_k (or $A_1^{(i)}, \dots, A_k^{(i)}$) in $[k]^{n-1}$ given by:

$$A_t = \{x = (x_1, \dots, x_{n-1}) \in [k]^{n-1} : (x_1, \dots, x_{i-1}, t, x_i, x_{i+1}, \dots, x_{n-1}) \in A\}.$$

The i -**compression** of A is the set $C_i(A) \subset [k]^n$ defined by:

$$(C_i(A)_t)_t \text{ is the initial segment of simplicial on } [k]^{n-1} \text{ of size } |A_t|.$$

Say A is i -**compressed** if $C_i(A) = A$.

Theorem 11. Let $A \subset [k]^n$, and let C be the initial segment of simplicial with $|C| = |A|$. Then $|N(C)| \leq |N(A)|$.

(Called the “vertex-isoperimetric inequality in the grid”.)

Proof. Induction on n . For $n = 1$: any $A \subset [k]^1$ with $A \neq \emptyset, [k]^1$, we have $|N(A)| \geq |A| + 1 = |N(C)|$.

Given $A \subset [k]^n$, fix $1 \leq i \leq n$.

Claim. $|N(C_i(A))| \leq |N(A)|$

Proof of claim. Write B for $C_i(A)$.

$$\text{For any } 1 \leq t \leq k, \text{ have } N(A)_t = N(A_t) \cup A_{t-1} \cup A_{t+1} \text{ (taking } A_0 = A_{k+1} = \emptyset).$$

$$\begin{array}{ccc} & \uparrow & \uparrow & \uparrow \\ & \text{own} & \text{below} & \text{above} \\ & \text{layer} & & \end{array}$$

$$\text{So } |N(A)| = \sum_t |N(A_t) \cup A_{t-1} \cup A_{t+1}|.$$

$$\text{Similarly, } |N(B)| = \sum_t |N(B_t) \cup B_{t-1} \cup B_{t+1}|.$$

But $|B_{t-1}| = |A_{t-1}|$, $|B_{t+1}| = |A_{t+1}|$, and $|N(B_t)| \leq |N(A_t)|$, by induction. Also, the sets $N(B_t)$, B_{t-1} , B_{t+1} are nested, as each is an initial segment of simplicial on $[k]^{n-1}$.

$$\text{Thus } |N(A_t) \cup A_{t-1} \cup A_{t+1}| \geq |N(B_t) \cup B_{t-1} \cup B_{t+1}|, \text{ and so } |N(A)| \geq |N(B)|.$$

Among all $B \subset [k]^n$ with $|B| = |A|$ and $|N(B)| \leq |N(A)|$, choose one for which $\sum_{x \in B}$ (position of x in simplicial) is minimal. Then B is i -compressed for all i (else $C_i(B)$ contradicts the minimality of B).

So it remains to show that $|N(B)| \geq |N(C)|$.

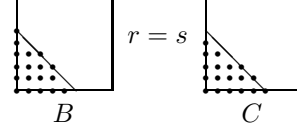
Case 1: $n = 2$.

B is i -compressed for all i iff B is a **down-set**. (I.e., if $x \in B$ and y has $y_i \leq x_i \forall i$, then $y \in B$.)

Let $s = \max\{|x| : x \in B\}$ and $r = \min\{|x| : x \notin B\}$.

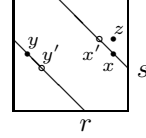
Then $r \leq s$ (else $B = C$).

If $r = s$, then $\{|x| : x < r\} \subset B \subset \{|x| : x \leq r\}$,
so certainly $|N(B)| \geq |N(C)|$.



If $r < s$, cannot have $\{x : |x| = s\} \subset B$, as B a down-set
(and $\exists x, |x| = r, x \notin B$). Hence there are x, x' such that
 $|x| = |x'| = s, x \in B, x' \notin B$, and $x = x' \pm (e_1 - e_2)$, where
 $e_i = (0, \dots, 0, 1, 0, \dots, 0)$.

Similarly, cannot have $\{y : |y| = r\} \subset B^c$, so again there are
 y, y' with $|y| = |y'| = r, y \in B, y' \notin B$, and $y = y' \pm (e_1 - e_2)$.



But now let $B' = B \cup \{x\} - \{y'\}$. Then $|N(B')| \leq |N(B)|$ (as we have lost ≥ 1 point
(e.g., z) and gained ≤ 1 point in the neighbourhood), contradicting the choice of B .

Case 2: $n \geq 3$.

For $x \in B$ with $x_n \geq 2$, have $x - e_n + e_i \in B$ for all i with $x_i < k$ (as B is j -
compressed, any $j \neq n, i$). Thus $N(B_t) \subset B_{t-1}$ for all $t = 2, \dots, k$. We had $N(B)_t =$
 $N(B_t) \cup B_{t+1} \cup B_{t-1}$. So actually have $N(B_t) = B_{t-1}$.

$$\begin{array}{ccccccc} \text{So } |N(B)| & = & |B_{k-1}| & + & |B_{k-2}| & + & \dots & + & |B_1| & + & |N(B_1)| & = & |B| - |B_k| + |N(B_1)|. \\ & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\ & & \text{height } k & & \text{height } k-1 & & & & \text{height } 2 & & \text{height } 1 & & \end{array}$$

Similarly, have $|N(C)| = |C| - |C_k| + |N(C_1)|$.

Thus, to complete the proof, it is sufficient to show that $|B_k| \leq |C_k|$ and $|B_1| \geq |C_1|$.

- $|B_k| \leq |C_k|$

Define a set $D \subset [k]^n$ by: $D_k = B_k$, and $D_t = N(D_{t+1})$ for $t = k-1, k-2, \dots, 1$.

Then D is an initial segment of simplicial, and $D \subset B$. So $|D| \leq |B| = |C|$, so $D \subset C$
(as D, C are initial segments of simplicial). Thus $D_k \subset C_k$, so $|D_k| \leq |C_k|$.

- $|B_1| \geq |C_1|$

Define a set $E \subset [k]^n$ by: $E_1 = B_1$, and $E_t = \{x \in [k]^{n-1} : N(\{x\}) \subset E_{t-1}\}$ for
 $t = 2, 3, \dots, k$.

Then E is an initial segment of simplicial, and $E \supset B$. So $|E| \geq |B| = |C|$, so $E \supset C$.
Thus $E_1 \supset C_1$, so $|E_1| \geq |C_1|$. \square

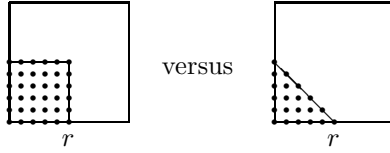
Corollary 12. Let $A \subset [k]^n$ with $|A| \geq |\{x : |x| \leq r\}|$. Then $|A_{(t)}| \geq |\{x : |x| \leq r + t\}|$. \square

Remark. Can check from this that for any fixed k , the sequence $([k]^n)_{n=1}^\infty$ is a normal Lévy family.

Edge-isoperimetric inequality in the grid

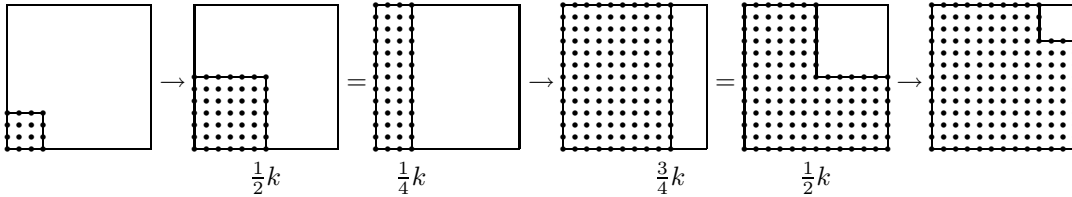
To minimise $|\partial A|$ for $|A|$ given.

$n = 2$.



First has size r^2 , second has size $\sim \frac{1}{2}r^2$, and both have boundary $2r$.

This suggests that squares are best. But:



Sadly, the extremal sets are *not* nested: there are “phase transitions” at size $k^2/4$ and $3k^2/4$.

$n = 3$. Get: cube $[a]^3 \rightarrow$ square column $[a]^2 \times [k] \rightarrow$ halfspace $[a] \times [k]^2$
 \rightarrow complement of square column \rightarrow complement of cube.

Aim. Best to take $[n]^d \times [k]^{n-d}$ (some d) or complements.

Observe that if $A = [a]^d \times [k]^{n-d}$ then $|\partial A| = da^{d-1}k^{n-d} = d|A|^{1-1/d}k^{n/d-1}$.

Theorem 13. Let $A \subset [k]^n$ with $|A| \leq k^n/2$.

Then $|\partial A| \geq \min \{d|A|^{1-1/d}k^{n/d-1} : d = 1, 2, \dots, n\}$.

(The “edge-isoperimetric inequality in the grid”.)

**** Non-examinable section ****

Proof (sketch). Wlog, A is a down-set in $[k]^n$. (Stamp on A – i.e., apply 1-dimensional compressions.)

For $1 \leq i \leq n$, define $C_i(A)$ by giving its i -sections:

$C_i(A)_t$ is extremal in $[k]^{n-1}$, with $|C_i(A)_t| = |A_t|$.
 \uparrow
 i.e., of form $[a]^d \times [k]^{n-1-d}$, or complement

Write B for $C_i(A)$.

Then $|\partial A| = \sum |\partial A_t| + |A_1| - |A_k|$.
 $\uparrow \quad \swarrow \quad \searrow$
 horizontal vertical, as A is a down-set

And $|\partial B| = \sum |\partial B_t| + ?$ – as extremal sets in $[k]^{n-1}$ are *not* nested.

Indeed, we can clearly have $|\partial B| > |\partial A|$.

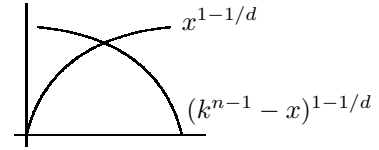
Try to introduce a ‘fake boundary’, ∂' . We would want $\partial' \leq \partial$, with equality for our extremal sets, and $\partial' C_i(A) \leq \partial' A$ for all A .

Obvious guess: $\partial' A = \sum |\partial A_t| + |A_1| - |A_k|$. (Note $\partial' = \partial$ on down-sets.) So we do have $\partial' B \leq \partial' A$, where $B = C_i(A)$. But it fails for C_j , $j \neq i$.

(Could try $\partial'' A = \sum_{i=1}^n |A_1^{(i)}| - |A_k^{(i)}|$ – cannot work, e.g. because the ‘outside shell’ of $[k]^n$ would have $\partial'' = 0$.)

We know: $|\partial A| = \partial' A \geq \partial' B = \sum |\partial B_t| + |B_1| - |B_k| = \sum f(|B_t|) + |B_1| - |B_k|$, where f is the extremal function in dimension $n - 1$.

Now, $f(x)$ is minimum of some functions of the form $c \cdot x^{1-1/d}$ and $c(k^{n-1} - x)^{1-1/d}$. Each of these is concave, so their pointwise minimum f is also concave.



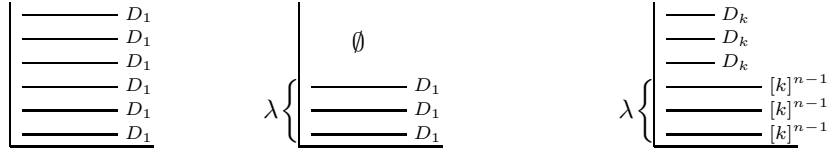
Consider varying $|B_2|, \dots, |B_{k-1}|$, keeping $|B_2| + \dots + |B_{k-1}|$ fixed, and such that $|B_1| \geq |B_2| \geq \dots \geq |B_{k-1}| \geq |B_k|$.

By concavity of f , have $\partial' B \geq \partial' C$, where for some λ we have $C_t = \begin{cases} B_1 & \text{if } t \leq \lambda \\ B_k & \text{if } t > \lambda \end{cases}$.

We have $|\partial A| = \partial' A \geq \partial' B \geq \partial' C$, with $\partial' C = \lambda f(|B_1|) + (k - \lambda) f(|B_k|) + |B_1| - |B_k|$ (but C still not a down-set).

Now consider varying $|B_1|$ and $|B_k|$ (λ fixed), keeping $\lambda|B_1| + (k - \lambda)|B_k|$ fixed, and keeping $|B_1| \geq |B_k|$. Again, this is a concave function of $|B_1|$ (as it is concave+concave+linear), so by concavity of f , we have $\partial' C \geq \partial' D$, where either

$$D_t = D_1 \quad \forall t, \quad \text{or} \quad D_t = \begin{cases} D_1 & \forall t \leq \lambda \\ \emptyset & \forall t > \lambda \end{cases}, \quad \text{or} \quad D_t = \begin{cases} [k]^{t-1} & \forall t \leq \lambda \\ D_k & \forall t > \lambda \end{cases}.$$



Thus $|\partial A| = \partial' A \geq \partial' B \geq \partial' C \geq \partial' D = \partial D$. (Miraculously, D is a down-set.)

So, take as our i -compression the map $A \mapsto D$.

Then finish proof as usual. □

Remark. To make the argument precise, work in the continuous cube $[0, 1]^n$ instead, and then pass to discrete at the end.

**** End of non-examinable section ****

Very few isoperimetric inequalities are known (even approximately).

For example, what about r -sets? Take the graph on $X^{(r)}$ with x, y joined if $|x \cap y| = r - 1$

(i.e., $d(x, y) = 2$ in Q_n). No good isoperimetric inequality is known.

Most interesting case is $r = n/2$. It's conjectured that balls are best – i.e., sets of the form $\{x \in X^{(r)} : d(x, x_0) \leq d\}$, i.e., of the form $\{x \subset [2r] : |x \cap [r]| \geq k\}$.

Unknown!

Chapter 3 : Intersecting Families

t -intersecting families

Say $A \subset \mathbb{P}(X)$ is t -**intersecting** if $|x \cap y| \geq t$ for all $x, y \in A$.

How large can A be?

E.g., $t = 2$. Could take $\{x : 1, 2 \in x\}$ – size $\frac{1}{4} \cdot 2^n$. Beaten by $\{x : |x| \geq \frac{n}{2} + 1\}$ – size $\sim \frac{1}{2} \cdot 2^n$.

Theorem 1 (Katona's t -intersecting theorem). Let $A \subset \mathbb{P}(X)$ be t -intersecting, where $n + t$ is even. Then $|A| \leq \left| X^{\left(\leq \frac{n+t}{2}\right)} \right|$.

Proof. For any $x, y \in A$, have $d(x, x^c) \geq t$. Thus writing $\bar{A} = \{x^c : x \in A\}$, we have $d(A, \bar{A}) \geq t$. I.e., $A_{(t-1)}$ is disjoint from \bar{A} .

Suppose $|A| > \left| X^{\left(\geq \frac{n+t}{2}\right)} \right|$. Then $|A_{(t-1)}| \geq \left| X^{\left(\geq \frac{n+t}{2} - (t-1)\right)} \right| = \left| X^{\left(\geq \frac{n-t}{2} + 1\right)} \right|$.
↑
by Harper

But $|\bar{A}| > \left| X^{\left(\leq \frac{n-t}{2}\right)} \right|$, contradicting $A_{(t-1)} \cap \bar{A} = \emptyset$. □

What about $A \subset X^{(r)}$ being t -intersecting? ($1 \leq t \leq r$)

Could guess $A_0 = \{x \in X^{(r)} : 1, 2, \dots, t \in x\}$.

Could also try, for $1 \leq \alpha \leq r - t$, the set $A_\alpha = \{x \in X^{(r)} : |x \cap \{1, 2, \dots, t + 2\alpha\}| \geq t + \alpha\}$.

E.g., $t = 2$, in $[7]^{(4)}$: $|A_0| = \binom{5}{2} = 10$, $|A_1| = 1 + \binom{4}{3} \binom{3}{1} = 13$, $|A_2| = \binom{6}{4} = 15$.

in $[8]^{(4)}$: $|A_0| = \binom{6}{2} = 15$, $|A_1| = 1 + \binom{4}{3} \binom{4}{1} = 17$, $|A_2| = \binom{6}{4} = 15$.

in $[9]^{(4)}$: $|A_0| = \binom{7}{2} = 21$, $|A_1| = 1 + \binom{4}{3} \binom{5}{1} = 21$, $|A_2| = \binom{6}{4} = 15$.

As $|A_0|$ is quadratic (in n), $|A_1|$ is linear, and $|A_2|$ is constant, we will have A_0 winning if n is large.

Theorem 2. Let $A \subset X^{(r)}$ be t -intersecting, where r, t are fixed, $1 \leq t \leq r$.

Then $|A| \leq |A_0| = \binom{n-t}{r-t}$, if n sufficiently large.

Remarks. 1. Sometimes called the “second Erdős-Ko-Rado Theorem”.

2. Bound on n is $(16r)^r$ (crude) or $2tr^3$ (careful).

Proof. “Idea: A_0 has $r - t$ degrees of freedom.”

Wlog A is maximal t -intersecting, and so there are $x, y \in A$ with $|x \cap y| = t$ (or else for all $x \in A$ and all $i \in x, j \notin x$, have $x \cup j - i \in A$ by maximality, hence $A = X^{(r)}$ – contradiction).

Cannot have $x \cap y \in z$ for all $z \in A$ (else $|A| \leq |A_0|$, so done). So choose $z \in A$ with $z \not\supseteq x \cap y$. Then for any $w \in A$, we must have $|w \cap (x \cup y \cup z)| \geq t + 1$.

$$\text{So } |A| \leq 2^{3r} \left(\binom{n}{r-t-1} + \binom{n}{r-t-2} + \dots + \binom{n}{0} \right),$$

\uparrow
 $w \text{ on } x \cup y \cup z$

\nwarrow
 $w \text{ off } x \cup y \cup z$

which is a polynomial of degree $r - t - 1$, hence $< |A_0|$ for n large. □

Remark. Frankl Conjecture was: $A \subset X^{(r)}$ r -intersecting $\Rightarrow |A| \leq \min(|A_0|, \dots, |A_{r-t}|)$ (any n). Proved by Ahlswede and Khachatrian in 1998.

Modular intersections

So far, we have banned $|x \cap y| = 0$. What if we instead banned $|x \cap y| \equiv 0 \pmod{\text{something}}$?

E.g., suppose r is odd, and we want $A \subset X^{(r)}$ with $|x \cap y|$ odd for all $x, y \in A$. We can achieve $|A| = \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{r-1}{2} \rfloor}$ by taking all sets x of the form: 1, together with $\frac{r-1}{2}$ of the pairs 23, 45, ...

What if (still for r odd) we wanted $|x \cap y|$ even for all $x, y \in A$ ($x \neq y$). We can achieve $n - r + 1$ by taking all x containing 1, 2, ... $r - 1$. Amazingly:

Proposition 3. Let r be odd and let $A \subset X^{(r)}$ have $|x \cap y|$ even for all $x, y \in A$ ($x \neq y$). Then $|A| \leq n$.

Idea. Try to find $|A|$ linearly independent vectors in an n -dimensional vector space.

Proof. View each point $x \in Q_n$ as a point \bar{x} in \mathbb{Z}_2^n , the n -dimensional vector space over the field \mathbb{Z}_2 . (E.g., $\{1, 3, 4\} \leftrightarrow 10110\dots 0$.) Consider $\{\bar{x} : x \in A\}$.

Have $\langle \bar{x}, \bar{x} \rangle = 1$ for all $x \in A$ (where $\langle \cdot, \cdot \rangle$ is the usual dot product), and $\langle \bar{x}, \bar{y} \rangle = 0$ for all $x, y \in A, x \neq y$. Hence $\{\bar{x} : x \in A\}$ is linearly independent. (If $\sum \lambda_i x_i = 0$, dot with x_i to get $\lambda_i = 0$.)

So $|\{\bar{x} : x \in A\}| \leq n$. □

For r even? If $|x \cap y|$ even for all distinct $x, y \in A \subset X^{(r)}$, we can have $|A| \geq \binom{\lfloor n/2 \rfloor}{r/2}$, by taking $r/2$ of the pairs 12, 34, ...

If $|x \cap y|$ is odd for all distinct $x, y \in A \subset X^{(r)}$, then we must have $|A| \leq n + 1$ – else add the point $n + 1$ to each $x \in A$ to contradict Proposition 3.

So banning $|x \cap y| \equiv r \pmod{2}$ forces $|A|$ to be small. Does this generalise?

We now show that “ s allowed intersections mod $p \Rightarrow |A| \leq \binom{n}{s}$ ”.

Theorem 4 (Frankl-Wilson Theorem). Let p be prime, and let $A \subset X^{(r)}$ be such that for all distinct $x, y \in A$ we have $|x \cap y| \equiv \lambda_i \pmod{p}$, some i , where $\lambda_1, \dots, \lambda_s$ ($s \leq r$) are integers, with none $\equiv r \pmod{p}$.

Then $|A| \leq \binom{n}{s}$.

Remarks. 1. The bound is a polynomial in n , independent of r (!)

2. Bound is essentially best possible. Take $A = \{x \in X^{(r)} : [r-s] \subset x\}$.

This has $|A| = \binom{n-r+s}{s} \sim \binom{n}{s}$. (Ratio $\rightarrow 1$ as $n \rightarrow \infty$.)

3. No polynomial bound if we allow $|x \cap y| \equiv r \pmod{p}$.

Indeed, write $r = a + \lambda p$ ($0 \leq a \leq p-1$). $\underbrace{(\leftarrow a \rightarrow)}_{\text{take this}} \underbrace{(\leftarrow p \rightarrow)(\leftarrow p \rightarrow)(\leftarrow p \rightarrow) \dots}_{\text{take } \lambda \text{ of these}}$

Can have $|x \cap y| \equiv r \pmod{p}$ for all $x, y \in A$, with $|A| = \binom{\lfloor \frac{n-a}{p} \rfloor}{\lambda}$. But the degree grows as λ grows.

Idea. Try to find $|A|$ linearly independent points in a vector space of dimension $\binom{n}{s}$, by somehow “applying the polynomial $(t - \lambda_1) \dots (t - \lambda_s)$ to the values of $|x \cap y|$ ”.

Proof. For $i \leq j$, let $M(i, j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix, with rows indexed by $X^{(i)}$ and columns indexed by $X^{(j)}$, with $M(i, j)_{xy} = \begin{cases} 1 & \text{if } x \subset y \\ 0 & \text{if } x \not\subset y \end{cases}$.

The matrix $M(s, r)$ has $\binom{n}{s}$ rows. Let V be the vector space (over \mathbb{R}) spanned by those rows. Thus $\dim V \leq \binom{n}{s}$.

Consider the matrix $M(i, s)M(s, r)$, where $i \leq s$. For $x \in X^{(i)}$ and $y \in X^{(r)}$,

$$\begin{aligned} (M(i, s)M(s, r))_{xy} &= \text{the number of } z \in X^{(s)} \text{ with } x \subset z \subset y \\ &= \begin{cases} \binom{r-i}{s-i} & \text{if } x \subset y \\ 0 & \text{otherwise} \end{cases} \\ &= \binom{r-i}{s-i} M(i, r)_{xy}. \end{aligned}$$

Thus $M(i, s)M(s, r) = \binom{r-i}{s-i} M(i, r)$, and the rows of $M(i, r)$ belong to V (as pre-multiplying is just taking linear combinations of the rows).

Consider $M(i) = M(i, r)^T M(i, r)$ – again, must have all rows in V . For $x, y \in X^{(r)}$,

$$\begin{aligned} M(i)_{xy} &= \text{the number of } z \in X^{(s)} \text{ with } z \subset x, z \subset y \\ &= \binom{|x \cap y|}{i}. \end{aligned}$$

Write the integer polynomial $(t - \lambda_1) \dots (t - \lambda_s)$ as $\sum_{i=0}^s a_i \binom{t}{i}$, for some integers a_i . (This is possible, as $t(t-1) \dots (t-i+1) = i! \binom{t}{i}$.)

Let $M = \sum_{i=0}^s a_i M(i)$. All rows are in V . Then

$$M_{xy} = \sum_{i=0}^s a_i \binom{|x \cap y|}{i} = (|x \cap y| - \lambda_1) \dots (|x \cap y| - \lambda_s)$$

$$\begin{cases} \equiv 0 \pmod{p} & \text{if } |x \cap y| \equiv \text{some } \lambda_i \pmod{p} \\ \not\equiv 0 \pmod{p} & \text{otherwise} \end{cases}$$

Now look at the submatrix whose rows and columns are indexed by A .

It is $\begin{pmatrix} \not\equiv 0 & & O \\ & \ddots & \\ O & & \not\equiv 0 \end{pmatrix}$.

Rows are integer-valued, and are linearly independent over \mathbb{Z}_p , so also over \mathbb{Z} , so also over \mathbb{Q} , so also over \mathbb{R} .

Thus $|A| \leq \dim V \leq \binom{n}{s}$. □

Remark. We do need p prime. Grolmusz constructed, for each n , a value of $r \equiv 0 \pmod{6}$ and $A \subset X^{(r)}$ with $|x \cap y| \not\equiv 0 \pmod{6}$ for all distinct $x, y \in A$, with $|A| > n^c \log n / \log \log n$, some $c > 0$. (Not a polynomial in n .)

Corollary 5. Let $A \subset X^{(r)}$ with $|x \cap y| \not\equiv r \pmod{p}$ for all distinct $x, y \in A$ (some prime $p < r$). Then $|A| \leq \binom{n}{p-1}$.

Proof. We are allowed $p-1$ values of $|x \cap y| \pmod{p}$. □

Two $\frac{n}{2}$ -sets from $[n]$ intersect (on average) in about $n/4$ points. But an intersection size of exactly $n/4$ is very unlikely. However, amazingly:

Corollary 6. Let p be prime, and let $A \subset [4p]^{(2p)}$ be such that $|x \cap y| \neq p$ for all $x, y \in A$.

Then $|A| \leq 2 \binom{4p}{p-1}$.

(“Not much of a constraint.”)

Note. $\binom{4p}{p-1}$ is extremely small: have $\binom{n}{n/4} \leq 2e^{-n/32} 2^n$, so $\binom{n}{n/4}$ is an exponentially small fraction of $\binom{n}{n/2} \sim \frac{c}{\sqrt{2}} 2^n$.

Proof. Halving $|A|$ is necessary, we may assume that we never have $x, x^c \in A$. Hence, for distinct $x, y \in A$, we have $|x \cap y| \neq 0, p$, so $|x \cap y| \not\equiv 0 \pmod{p}$.

Thus $|A| \leq \binom{4p}{p-1}$ by Corollary 5. □

Borsuk's Conjecture

Suppose S is a bounded set in \mathbb{R}^n . How many pieces do we need to break S into such that each piece has smaller diameter than S ? Taking $S \subset \mathbb{R}^n$ to be a regular simplex (i.e., $n + 1$ equally-spaced points), it's clear that we may need as many as $n + 1$ pieces.

Borsuk's Conjecture. $n + 1$ pieces is always possible.

Known for $n = 1, 2, 3$, and also for $S \subset \mathbb{R}^n$ being a smooth convex body or a symmetric convex body (i.e., $x \in S \Rightarrow -x \in S$).

In fact, Borsuk's Conjecture is massively false.

Theorem 7 (Kahn, Kalai). For all n , there exists a bounded set $S \subset \mathbb{R}^n$ such that to break S into pieces of smaller diameter, we need $\geq c\sqrt[n]{n}$ pieces, for some constant $c > 1$.

Notes. 1. Our proof will give Borsuk false for $n \geq 2000$.

2. We'll prove Theorem 7 for n of the form $\binom{4p}{2}$, p prime – then it follows (with a different c) for all n , e.g. because there exists p with $n/2 \leq p \leq n$.

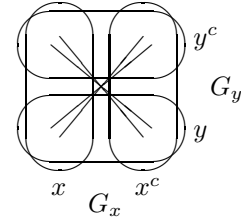
Proof. We'll consider $S \subset \mathbb{Q}^n \subset \mathbb{R}^n$. In fact $S \subset [n]^{(r)}$ (“A genuine idea.”)

For $x, y \in S$, we have $\|x - y\|^2 = 2(r - |x \cap y|)$ (where $\|\cdot\|$ is the usual Euclidean distance). Thus we seek S with $\min |x \cap y| = k$ say, but any subset of S with $\min |x \cap y| > k$ is much smaller than S .

Let $n = \binom{4p}{2}$, where p is prime. Identify $[n]$ with $E(K_{4p})$ – the edge-set of the complete graph on $\{1, \dots, 4p\}$. For each $x \in [4p]^{\binom{4p}{2}}$, let G_x be the complete bipartite graph on vertex classes x, x^c .

Let $S = \{G_x : x \in [4p]^{\binom{4p}{2}}\}$. So $S \subset [n]^{\binom{4p}{2}}$, and $|S| = \frac{1}{2} \binom{4p}{2}$.

$$\begin{aligned} \text{We have } |G_x \cap G_y| &= |x \cap y|^2 + (2p - |x \cap y|)^2 \\ &= d^2 + (2p - d)^2, \text{ where } d = |x \cap y|. \end{aligned}$$



This is minimal when $d = p$, i.e. when $|x \cap y| = p$.

Now let $S' \subset S$ have smaller diameter than S . Say $S' = \{G_x : x \in A\}$.

Then we must have $|x \cap y| \neq p$ for all $x, y \in A$ (else diameter of $S' =$ diameter of S).

So $|A| \leq 2 \binom{4p}{2p-1}$, by Corollary 6. Thus the number of pieces needed is

$$\begin{aligned} &\geq \frac{\frac{1}{2} \binom{4p}{2p}}{2 \binom{4p}{2p-1}} \geq \frac{c 2^{4p} / \sqrt{p}}{4e^{-p/8} 2^{4p}} \geq (c')^p \text{ (some } c' > 1) \\ &\geq (c'')^{\sqrt{n}} \text{ (some } c'' > 1). \end{aligned}$$

□

Notes typeset in L^AT_EX by Gareth Taylor

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1. Write down two antichains of size 10 in $\mathcal{P}([5])$. Write down an antichain of size 8 in $\mathcal{P}([5])$ whose members do not all have the same size.
2. What are the 99th, 100th and 101st elements in the colex order on $\mathbb{N}^{(4)}$? For which $A \in \mathbb{N}^{(4)}$ is it true that A and the successor of A (in colex) have the same sum?
3. Let $\mathcal{A} \subset [9]^{(3)}$ with $|\mathcal{A}| = 28$. How small can the lower shadow of \mathcal{A} be? And the upper shadow?
4. Let n be even, and let $\mathcal{A} \subset \mathcal{P}(X)$ be a set system that contains no chain of length 3. Prove that $|\mathcal{A}| \leq \binom{n}{n/2} + \binom{n}{n/2 - 1}$.
5. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an antichain not of the form $X^{(r)}$, $0 \leq r \leq n$. Must there exist a maximal chain that is disjoint from \mathcal{A} ?
6. A set system $\mathcal{A} \subset \mathcal{P}(X)$ is called a *cross-cut* if for every $B \in \mathcal{P}(X)$ there exists $A \in \mathcal{A}$ with $B \subset A$ or $A \subset B$. Prove that every cross-cut contains a cross-cut of size at most $\binom{n}{\lfloor n/2 \rfloor}$. Does every cross-cut contain a cross-cut that is an antichain?
7. Let $\mathcal{A} \subset X^{(r)}$, and let $U, V \subset X$ with $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. If \mathcal{A} is left-compressed, can we have $|\partial C_{UV}(\mathcal{A})| > |\partial \mathcal{A}|$?
8. Find a set system \mathcal{A} for which equality holds in the Kruskal-Katona theorem but which is not isomorphic to an initial segment of colex.
9. Let x_1, \dots, x_n be non-zero real numbers, and let a be real. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\sum_{i \in A} x_i$, $A \subset [n]$, can be equal to a .
- +10. For $n = 2r + 1$, give an explicit bijection $f : X^{(r)} \rightarrow X^{(r+1)}$ such that $A \subset f(A)$ for every $A \in X^{(r)}$.
- +11. Let $\mathcal{A} \subset X^{(r)}$ and $\mathcal{B} \subset X^{(r+1)}$ be initial segments of colex with $|\mathcal{A}| = |\mathcal{B}|$. Do we always have $|\partial \mathcal{A}| \leq |\partial \mathcal{B}|$?

1. Let $r < n/2$. What is the largest intersecting family contained in $X^{(\leq r)}$?
2. A set system $A \subset \mathcal{P}(X)$ is an *up-set* if whenever $x \in A$ and $x \subset y$ then also $y \in A$. Explain why every maximal intersecting family is an up-set of size 2^{n-1} . Conversely, if A is an up-set with $|A| = 2^{n-1}$, must A be intersecting?
3. Let $A \subset \mathcal{P}(X)$ with $|A| = 2^{n-1} + 1$, so that some pair $x, y \in A$ must be disjoint. What is the smallest number of such disjoint pairs that A can have? And what if $|A| = 2^{n-1} + 2$?
4. Let $A \subset Q_6$ with $|A| = 26$. How small can the vertex-boundary of A be? And the edge-boundary?
5. Suppose that we try to use codimension-1 compressions to prove the (false) result that initial segments of the simplicial order on Q_n minimise the edge-boundary. Where does the proof go wrong?
6. Write down a direct proof of the Kruskal-Katona theorem using codimension-1 compressions.
7. Does the sequence of paths P_1, P_2, P_3, \dots form a Lévy family? What about the sequence of two-dimensional grids $[1]^2, [2]^2, [3]^2, \dots$?
8. Give an example of a connected graph G for which the extremal sets for the vertex-isoperimetric inequality do not form a nested family: in other words, there is no ordering of the vertices of G such that to minimise the neighbourhood of a set of m vertices it is best to take the first m vertices in that ordering.
9. Let f_1, \dots, f_{n^2} be Lipschitz functions on Q_n . Prove that, for n sufficiently large, there exists a point $x \in Q_n$ such that $|f_i(x) - f_i(x^c)| < n/100$ for all i .
10. Let $A \subset Q_n$ with $|A| = |X^{(\leq r)}|$, where $r < n/2$. Prove that we can always find a set of $\binom{n}{r}$ disjoint edges between A and A^c .
11. Let $A_1, A_2, \dots, A_d \subset \mathcal{P}(X)$ be intersecting families. Prove that $|A_1 \cup A_2 \cup \dots \cup A_d| \leq 2^n - 2^{n-d}$.
12. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ be an intersecting family of finite sets. Must there exist a finite set $F \subset \mathbb{N}$ such that the family $\{A \cap F : A \in \mathcal{A}\}$ is intersecting? And what if $\mathcal{A} \subset \mathbb{N}^{(r)}$?

1. Let $A \subset [4]^3$ with $|A| = 23$. How small can the vertex-boundary of A be?
2. The *distance* between two (non-empty) subsets A and B of $[k]^n$ is $d(A, B) = \min \{d(x, y) : x \in A, y \in B\}$. Show that if $d(A, B) \geq t$ then the distance between the first $|A|$ and the last $|B|$ points of $[k]^n$ in the simplicial order is also at least t .
3. Suppose that we wish to remove edges from $[k]^n$ in such a way that the resulting graph has all components of size at most $k^n/2$. Show that we must remove at least k^{n-1} edges.
4. The *diameter* of a set $A \subset Q_n$ is $\max \{d(x, y) : x, y \in A\}$. Deduce from Harper's theorem that if $A \subset Q_n$ has diameter d , where $d < n$ and d is even, then $|A| \leq |X^{(\leq d/2)}|$.
5. Let r be an odd positive integer. Show that there are infinitely many values of n for which there exists a set system $A \subset [n]^{(r)}$ satisfying $|x \cap y|$ even for all distinct $x, y \in A$ and having size $|A| = n$.
6. Does there exist an even positive integer r and a value of n greater than $r + 1$ for which there exists a set system $A \subset [n]^{(r)}$ satisfying $|x \cap y|$ odd for all distinct $x, y \in A$ and having size $|A| = n$?
7. Let $A \subset [n]^{(r)}$ be such that the number of values taken by $|x \cap y|$, over all distinct $x, y \in A$, is s . Prove that $|A| \leq \binom{n}{s}$.
8. Give a constructive proof that the Ramsey number $R(t)$ grows faster than any polynomial in t , by considering the graph on $[p^3]^{(p^2-1)}$, where p is prime, in which x is joined to y if $|x \cap y|$ is congruent to $-1 \pmod p$.
9. Consider the graph on \mathbb{R}^n in which x is joined to y if $\|x - y\| = 1$, where $\|\cdot\|$ denotes the usual Euclidean distance. Show that this graph has chromatic number at least c^n , for some constant $c > 1$.
- +10. Let A be a subset of $[k]^n$ of diameter d , where k is odd and d is even. How large can A be?