Representation Theory

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REPRESENTATION THEORY (D) **24 lectures, Lent term**

Linear Algebra, and Groups, Rings and Modules are essential.

Representations of finite groups

Representations of groups on vector spaces, matrix representations. Equivalence of representations. Invariant subspaces and submodules. Irreducibility and Schur's Lemma. Complete reducibility for finite groups. Irreducible representations of Abelian groups.

Characters

Determination of a representation by its character. The group algebra, conjugacy classes, and or- thogonality relations. Regular representation. Induced representations and the Frobenius reciprocity theorem. Mackey's theorem. [12]

Arithmetic properties of characters

Divisibility of the order of the group by the degrees of its irreducible characters. Burnside's $p^a q^b$ theorem. [2]

Tensor products

Tensor products of representations. The character ring. Tensor, symmetric and exterior algebras. [3]

Representations of S^1 and SU_2

The groups S^1 and SU_2 , their irreducible representations, complete reducibility. The Clebsch-Gordan formula. *Compact groups.* [4]

Further worked examples

The characters of one of $GL_2(F_q)$, S_n or the Heisenberg group. [3]

Appropriate books

J.L. Alperin and R.B. Bell *Groups and representations.* Springer 1995 (£37.50 paperback).
I.M. Isaacs *Character theory of finite groups.* Dover Publications 1994 (£12.95 paperback).
G.D. James and M.W. Liebeck *Representations and characters of groups.* Second Edition, CUP 2001 (£24.99 paperback).
J-P. Serre *Linear representations of finite groups.* Springer-Verlag 1977 (£42.50 hardback).
M. Artin *Algebra.* Prentice Hall 1991 (£56.99 hardback).

Representation Theory

This is the theory of how groups act as groups of transformations on vector spaces.

• group (usually) means finite group.
• vector spaces are finite-dimensional and (usually) over $\mathbb{C}$.

1. Group Actions

• F a field – usually $F = \mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$: **ordinary** representation theory;
– sometimes $F = \mathbb{F}_p$ or $\overline{\mathbb{F}_p}$ (algebraic closure) : **modular** representation theory.

• V a vector space over F – always finite-dimensional over F.

• $GL(V) = \{\theta : V \to V,\, \theta \text{ linear, invertible}\}$ – group operation is composition, identity is 1.

Basic linear algebra

If $\dim_F V = n < \infty$, choose a basis $e_1, \ldots, e_n$ over F so that we can identify it with F^n. Then $\theta \in GL(V)$ corresponds to a matrix $A_\theta = (a_{ij}) \in F_{n\times n}$ where $\theta(e_j) = \sum_i a_{ij} e_i$, and $A_\theta \in GL_n(F)$, the **general linear** group.

(1.1) $GL(V) \cong GL_n(F)$, $\theta \mapsto A_\theta$. (A group isomorphism – check $A_{\theta_1\theta_2} = A_{\theta_1}A_{\theta_2}$, bijection.)

Choosing different bases gives different isomorphisms to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ with respect to different bases iff they are **conjugate/similar**, viz. there exists $X \in GL_n(F)$ such that $A_2 = XA_1X^{-1}$.

Recall the **trace** of A, $\mathrm{tr}\,(A) = \sum_i a_{ii}$ where $A = (a_{ij}) \in F_{n\times n}$.

(1.3) $\mathrm{tr}\,(XAX^{-1}) = \mathrm{tr}\,(A)$, hence define $\mathrm{tr}\,(\theta) = \mathrm{tr}\,(A)$, independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V is finite-dimensional over $\mathbb{C}$ and α is **idempotent**, i.e. $\alpha^m = \mathrm{id}$, some m. Then α is diagonalisable. (Proof uses Jordan blocks – see Telemann p.4.)

Recall $\mathrm{End}(V)$, the **endomorphism algebra**, is the set of all linear maps $V \to V$ with natural addition of linear maps, and the composition as 'multiplication'.

(1.5) Proposition. Take V finite-dimensional over $\mathbb{C}$, $\alpha \in \mathrm{End}(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors such that $f(\alpha) = 0$.

Recall in (1.4), $\alpha^m = \mathrm{id}$, so take $f = X^m - 1 = \prod_{j=0}^{m-1}(X - \omega^j)$, where $\omega = e^{2\pi i/n}$.

Proof of (1.5). $f(X) = (X - \lambda_1)\ldots(X - \lambda_k)$.

$$\text{Let } f_j(X) = \frac{(X-\lambda_1)\ldots\cancel{(X-\lambda_j)}\ldots(X-\lambda_k)}{(\lambda_j-\lambda_1)\ldots\cancel{(\lambda_j-\lambda_j)}\ldots(\lambda_j-\lambda_k)}.$$

So $1 = \sum f_j(X)$. Put $V_j = f_j(\alpha)V$. The $f_j(\alpha)$ are orthogonal projections, and $V = \bigoplus V_j$ with $V_j \subseteq V(\lambda_j)$ the λ_k-eigenspace. $\square$

(1.4*) In fact, a finite family of commuting separately diagonalisable automorphisms of a $\mathbb{C}$-space can be simultaneously diagonalised.

Basic group theory

(1.6) Symmetric group, $S_n = \mathrm{Sym}(X_n)$ on the set $X_n = \{1, \ldots, n\}$, is the set of all **permutations** (bijections $X_n \to X_n$) of X_n. $|S_n| = n!$

Alternating group, A_n on X_n, is the set of products of an even number of transpositions $(ij) \in S_n$. ('A_n is mysterious. Results true for S_n usually fail for A_n!')

(1.7) Cyclic group of order n, $C_n = \langle x : x^n = 1 \rangle$. E.g., $\mathbb{Z}/n\mathbb{Z}$ under $+$.

It's also the group of rotations, centre 0, of the regular n-gon in $\mathbb{R}^2$. And also the group of n^{th} roots of unity in $\mathbb{C}$ (living in $GL_1(\mathbb{C})$).

(1.8) Dihedral group, D_{2m} of order $2m$. $D_{2m} = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$.

Can think of this as the set of rotations and reflections preserving a regular m-gon (living in $GL_2(\mathbb{R})$). E.g., D_8, of the square.

(1.9) Quaternion group, $Q_8 = \langle x, y : x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$ of order 8.

('Often used as a counterexample to dihedral results.')

In $GL_2(\mathbb{C})$, can put $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(1.10) The **conjugacy class** of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$.

Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$ is the **centraliser** of $g \in G$.

Definition. G a group, X a set. G **acts on** X if there exists a map $* : G \times X \to X$, $(g, x) \mapsto g * x$, written gx for $g \in G$, $x \in X$, such that:

$$\begin{aligned} 1x &= x && \text{for all } x \in X \\ (gh)x &= g(hx) && \text{for all } g, h \in G, x \in X. \end{aligned}$$

Given an action of G on X, we obtain a homomorphism $\theta : G \to \mathrm{Sym}(X)$ called the **permutation representation** of G.

Proof. For $g \in G$, the function $\theta_g : X \to X$, $x \mapsto gx$, is a permutation (inverse is $\theta_{g^{-1}}$). Moreover, for all $g_1, g_2 \in G$, $\theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2}$ since $(g_1 g_2)x = g_1(g_2 x)$ for $x \in X$. □

In this course, X is often a finite-dimensional vector space, and the action is **linear**, viz: $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv$ for all $v, v_1, v_2 \in V = X$, $g \in G$, $\lambda \in F$.

2. Linear Representations

G a finite group. F a field, usually $\mathbb{C}$.

(2.1) Definition. Let V be a finite-dimensional vector space over F. A **(linear) representation** of G on V is a homomorphism $\rho = \rho_V : G \to GL(V)$.

We write ρ_g for $\rho_V(g)$. So for each $g \in G$, $\rho_g \in GL(V)$ and $\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}$.

The **dimension** or **degree** of ρ is $\dim_F V$.

(2.2) Recall $\ker\rho \lhd G$ and $G/\ker\rho \cong \rho(G) \leqslant GL(V)$. (The first isomorphism theorem.) We say that ρ is **faithful** if $\ker\rho = 1$.

Alternative (and equivalent) approach:

(2.3) G **acts linearly** on V if there exists a linear action $G \times V \to V$, viz:

$$\begin{aligned} \text{action: } & (g_1g_2)v = g_1(g_2v),\ 1v = v, \text{ for all } g_1, g_2 \in G,\ v \in V \\ \text{linearity: } & g(v_1 + v_2) = gv_1 + gv_2,\ g(\lambda v) = \lambda gv, \text{ for all } g \in G,\ v \in V,\ \lambda \in F. \end{aligned}$$

So if G acts linearly on V, the map $G \to GL(V)$, $g \mapsto \rho_g$, with $\rho_g : v \mapsto gv$, is a **representation** of V. And conversely, given a representation $G \to GL(V)$, we have a linear action of G on V via $g \cdot v = \rho(g)v$, for all $v \in V$, $g \in G$.

(2.4) In (2.3) we also say that V is a G**-space** or a G**-module**. In fact, if we define the **group algebra** $FG = \left\{ \sum_{g\in G} \alpha_g g : \alpha_g \in F \right\}$ then V is actually an FG-module.

Closely related:

(2.5) R is a **matrix representation** of G of degree n if R is a homomorphism $G \to GL_n(F)$.

Given a linear representation $\rho : G \to GL(V)$ with $\dim_F V = n$, fix basis $\mathcal{B}$; get a matrix representation $G \to GL_n(F)$, $g \mapsto [\rho(g)]_{\mathcal{B}}$.

Conversely, given matrix representation $R : G \to GL_n(F)$, we get a linear representation $\rho : G \to GL(V)$, $g \mapsto \rho_g$, via $\rho_g(v) = R_g(v)$.

(2.6) Example. Given any group G, take $V = F$ (the 1-dimensional space) and $\rho : G \to GL(V)$, $g \mapsto (\mathrm{id} : F \to F)$. This is known as the **trivial/principal** representation. So $\deg\rho = 1$.

(2.7) Example. $G = C_4 = \langle x : x^4 = 1\rangle$.

Let $n = 2$ and $F = \mathbb{C}$. Then $R : x \mapsto X$ (some matrix X) will determine all $x^j \mapsto X^j$. We need $X^4 = I$. We can take X diagonal with diagonal entries $\in \{\pm 1, \pm i\}$ (16 choices). Or we can take X not diagonal, then it will be isomorphic to some diagonal matrix, by (1.4).

(2.8) Definition. Fix G, F. Let V, V' be F-spaces and $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ be representations of G. The linear map $\phi : V \to V'$ is a G**-homomorphism** if

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \phi\downarrow & & \downarrow\phi \\ V' & \xrightarrow[\rho'_g]{} & V' \end{array}$$

the square commutes

$$(*) \quad \phi\rho(g) = \rho'(g)\phi \text{ for all } g \in G.$$

We say ϕ **intertwines** ρ, ρ'.

We write $\mathrm{Hom}_G(V, V')$ for the F-space of all of these.

We say that ϕ is a G-**isomorphism** if also ϕ is bijective; if such a ϕ exists we say that ρ, ρ' are **isomorphic**. If ϕ is a G-isomorphism, we write $(*)$ as $\rho' = \phi\rho\phi^{-1}$ (meaning $\rho'(g) = \phi\rho(g)\phi^{-1}$ for all $g \in G$).

(2.9) The relation of being isomorphic is an equivalence relation on the set of all linear representations of G (over F).

Remark. The basic problem of representation theory is to classify all representations of a given group G up to isomorphisms. Good theory exists for finite groups over $\mathbb{C}$, and for compact topological groups.

(2.10) If ρ, ρ' are isomorphic representations, they have the same dimension. Converse is false: in C_4 there are four non-isomorphic 1-dimensional representations. If $\omega = e^{2\pi i/4}$ then we have $\rho_j(\omega^i) = \omega^{ij}$ $(0 \leqslant i \leqslant 3)$.

(2.11) Given G, V over F of dimension n and $\rho : G \to GL(V)$. Fix a basis $\mathcal{B}$ for V; we get a linear isomorphism $\phi : V \to F^n$, $v \mapsto [v]_{\mathcal{B}}$. Get a representation $\rho' : G \to GL(F^n)$ isomorphic to ρ.

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\phi"'] & V \arrow[d, "\phi"] \\ F^n \arrow[r, "\rho'"'] & F^n \end{tikzcd}$$

(2.12) In terms of matrix representations, $R : G \to GL_n(F)$, $R' : G \to GL_n(F)$ are G-**isomorphic** if there exists a (non-singular) matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1}$ (for all $g \in G$).

In terms of G-actions, the actions of G on V, V' are G-isomorphic if there is an isomorphism $\phi : V \to V'$ such that $g\underbrace{\phi(v)}_{\text{in } V'} = \phi\underbrace{(gv)}_{\text{in } V}$ for all $g \in G, v \in V$.

Subrepresentations

(2.13) Let $\rho : G \to GL(V)$ be a representation of G. Say that $W \leqslant V$ is a G-**subspace** if it's a subspace and is $\rho(G)$-invariant, i.e. $\rho_g(W) \subseteq W$ for all $g \in G$. E.g., $\{0\}$ and V.

Say ρ is **irreducible**, or **simple**, if there is no proper G-subspace.

(2.14) Example. Any 1-dimensional representation of G is irreducible. (But not conversely: e.g. D_6 has a 2-dimensional $\mathbb{C}$-irreducible representation.)

(2.15) In definition (2.13) if W is a G-subspace then the corresponding map $G \to GL(W)$, $g \mapsto \rho(g)|_W$ is a representation of G, a **subrepresentation** of ρ.

(2.16) Lemma. $\rho : G \to GL(V)$ a representation. If W is a G-subspace of V and if $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of V containing the basis $\{v_1, \ldots, v_m\}$ of W, then the matrix of $\rho(g)$ with respect to $\mathcal{B}$ is (with the top-left $*$ being $m \times m$)

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \text{(for each } g \in G)$$

(2.17) Examples

(i) (2.10) revisited. The irreducible representations of $C_4 = \langle x : x^4 = 1\rangle$ are all 1-dimensional, and four of these are $x \mapsto i$, $x \mapsto -1$, $x \mapsto -i$, $x \mapsto 1$.

(The two $x \mapsto \pm i$ are faithful, since they have trivial kernel.)

In general, $C_m = \langle x : x^m = 1\rangle$ has precisely m irreducible complex representations, all of degree 1. Put $\omega = e^{2\pi i/m} \in \mu_m$ and define ρ_k by $\rho_k : x^j \mapsto \omega^{jk}$ $(0 \leqslant j, k \leqslant m-1)$.

It turns out that all irreducible complex representations of a finite abelian group are 1-dimensional: (1.4*) or see (4.4) below.

(ii) $G = D_6 = \langle x, y : x^3 = y^2 = 1, yxy^{-1} = x^{-1}\rangle$, the smallest non-abelian finite group. $G \cong S_3$ (generated by a 3-cycle and a 2-cycle).

G has the following irreducible complex representations:

$$\begin{array}{ll} \text{2 of degree 1}: & \rho_1 : x \mapsto 1, y \mapsto 1 \\ & \rho_2 : x \mapsto 1, y \mapsto -1 \\ \text{1 of degree 2}: & \rho_3 : x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ where } \omega = e^{2\pi i/3} \in \mu_3 \end{array}$$

This follows easily later on. For now, by brute force...

$$\begin{array}{lll} \text{Define} & u_0 = 1 + x + x^2, & v_0 = u_0 y, \\ & u_1 = 1 + \omega^2 x + \omega x^2, & v_1 = u_1 y, \\ & u_2 = 1 + \omega x + \omega^2 x^2, & v_2 = u_2 y. \end{array}$$

Check easily $xu_1 = x + \omega^2 x^2 + \omega = \omega u_1$, and in general $xu_i = \omega^i u_i$ $(0 \leqslant i \leqslant 2)$. (I.e., in the action of x, u_i is an eigenvector, of eigenvalue ω^i.) So $\langle u_i\rangle$, $\langle v_i\rangle$ are $\mathbb{C}\langle x\rangle$-modules.

$$\text{Also:}\quad \left.\begin{array}{ll} yu_0 = v_0, & yv_0 = u_0, \\ yu_1 = v_2, & yv_1 = u_2, \\ yu_2 = v_1, & yv_2 = u_1. \end{array}\right\} \quad \begin{array}{l} \text{So } \langle u_0, v_0\rangle, \langle u_1, v_2\rangle, \langle u_2, v_1\rangle \text{ are } \mathbb{C}\langle y\rangle\text{-modules,} \\ \text{and hence are all } \mathbb{C}G\text{-submodules.} \end{array}$$

Note, $U_3 = \langle u_1, v_2\rangle$, $U_4 = \langle u_2, v_1\rangle$ are irreducible and $\langle u_0, v_0\rangle$ has $U_1 = \langle u_0 + v_0\rangle$ and $U_2 = \langle u_0 - v_0\rangle$ as $\mathbb{C}G$-submodules.

Moreover, $\mathbb{C}D_6 = U_1 \oplus U_2 \oplus \underbrace{U_3 \oplus U_4}$.

$$\begin{array}{ccc} \nearrow & \uparrow & \nwarrow \\ \text{trivial} & \text{non-trivial} & \text{isomorphic via } u_1 \mapsto v_1,\ v_2 \mapsto u_2 \end{array}$$

(iii) $G = D_8 = \langle x, y : x^4 = y^2 = 1, yxy^{-1} = x^{-1}\rangle$, the rotations/reflections of a square. $G \lesssim S_4$. See (1.8) with $m = 4$.

G has the following irreducible complex representations:

$$\begin{array}{ll} \text{4 of degree 1}: & \rho_0 :\ x \mapsto 1,\ y \mapsto 1 \ \text{(trivial)} \\ & \rho_1 :\ x \mapsto 1,\ y \mapsto -1 \\ & \rho_2 :\ x \mapsto -1,\ y \mapsto 1 \\ & \rho_3 :\ x \mapsto -1,\ y \mapsto -1 \\ \text{1 of degree 2}: & \rho_4 : x \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \text{(up to isomorphism)} \end{array}$$

By considering the effect of y on eigenvectors of x, we'll show that any irreducible representation of G is isomorphic to one of the ρ_i. This is easy to do later. Here, let V be some *irreducible* G-space.

Under the action of x, we have

$$V|_x = V_1 \oplus V_{-1} \oplus V_i \oplus V_{-i}$$

where $V_\lambda = \{v \in V : xv = \lambda v\}$.

For the y-action: if $xv = v$ then $yv \in V_1$, since $x(yv) = yx^{-1}v = yv$; similarly if $xv = -v$ then $yv \in V_{-1}$, since $x(yv) = yx^{-1}v = -yv$.

So, if V is irreducible and $V_1 \neq 0$ or $V_{-1} \neq 0$, then V is 1-dimensional (so one of the ρ_j, $0 \leqslant j \leqslant 3$).

$$\left.\begin{array}{l}\text{Taking } v \in V_1, \text{ we have } xv = v, \text{ and } yv \text{ is either } v \text{ or } -v.\\ \text{Taking } v \in V_{-1}, \text{ we have } xv = -v, \text{ and } yv \text{ is either } v \text{ or } -v.\end{array}\right\} \quad \text{So, four cases.}$$

Final case: $V = V_i \oplus V_{-i}$.

Let $v \in V_i$, i.e. $xv = iv$. Then $yv \in V_{-i}$, since $x(yv) = yx^{-1}v = -iyv$, and vice versa.

Clearly $\langle v, yv\rangle$ is G-invariant, so $V = \langle v, yv\rangle$ as V is irreducible. Taking basis $\{v, yv\}$ we have x, y acting as in ρ_4 (with respect to this basis).

See James & Liebeck, p. 94, and also Example Sheet 1.

(2.18) Definition. We say that $\rho : G \to GL(V)$ is **decomposable** if there are G-invariant subspaces U, W with $V = U \oplus W$. Say ρ is a **direct sum** $\rho_U \oplus \rho_W$. If no such exists, we say ρ is **indecomposable**.

(U, W must have G-actions on them, not just ordinary vector subspaces.)

(2.19) Lemma. Suppose $\rho : G \to GL(V)$ is a decomposition with G-invariant decomposition $V = U \oplus W$. If $\mathcal{B}$ is a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_\ell\}$ consisting of a basis $\mathcal{B}_1$ of U and $\mathcal{B}_2$ of W, then with respect to $\mathcal{B}$,

$$\rho(g)_{\mathcal{B}} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} = \begin{bmatrix} [\rho_U(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_W(g)]_{\mathcal{B}_2} \end{bmatrix}$$

(2.20) Definition. $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$. The **direct sum** of ρ, ρ' is

$$\rho \oplus \rho' : G \to GL(V \oplus V'), \quad (\rho \oplus \rho')(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$$

– a block diagonal action.

For matrix representations, $R : G \to GL_n(F)$, $R' : G \to GL_{n'}(F)$, define

$$R \oplus R' : G \to GL_{n+n'}(F), \quad g \mapsto \begin{bmatrix} R(g) & 0 \\ 0 & R'(g) \end{bmatrix}, \quad \forall g \in G.$$

3. Complete Reducibility and Maschke's Theorem

G, F as usual.

(3.1) Definition. The representation $\rho : G \to GL(V)$ is **completely reducible**, or **semisimple**, if it is a direct sum of irreducible representations.

Evidently, simple $\Rightarrow$ completely reducible, but not conversely.

(3.2) Examples. Not all representations are completely reducible.

(i) $G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$, $V = \mathbb{C}^2$, natural action (gv is matrix multiplication).

V is not completely reducible. (Note G not finite.)

(ii) $G = C_p$, $F = \mathbb{F}_p$. $x^j \mapsto \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}$ ($0 \leqslant j \leqslant p-1$) defines a representation $G \to GL_2(F)$.

$V = \langle v_1, v_2 \rangle$ where $x^j v_1 = v_1$, $x^j v_2 = jv_1 + v_2$. Define $W = \langle v_1 \rangle$.

Then W is an FC_p-module but there is no X s.t. $V = W \oplus X$. (Note $F \neq \mathbb{R}, \mathbb{C}$.)

(3.3) Theorem (Complete Reducibility Theorem). Every finite-dimensional representation of a finite group over a field of characteristic 0 is completely reducible.

Enough to prove the following.

(3.4) Theorem (Maschke's Theorem). G finite, $\rho : G \to GL(V)$ with V an F-space, char $F = 0$. If W is a G-subspace of V then there exists a G-subspace U of V such that $V = W \oplus U$ (a direct sum of G-subspaces).

Note. The proof below also works for $(\operatorname{char} F, |G|) = 1$.

Proof 1. Let W' be any vector space complement of W in V, i.e. $V = W \oplus W'$. Let $q : V \to W$ be the projection of V onto W along W', i.e. if $v = w + w'$ then $q(v) = w$.

Define $\overline{q} : v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g) q(\rho(g^{-1})v)$, the 'average of q over G'.

Drop the ρs – i.e. write $\rho(g)q(\rho(g^{-1})v)$ as $gq(g^{-1}v)$.

Claim (i). $\overline{q} : V \to W$.

For $v \in V$, $q(g^{-1}v) \in W$ and $gW \subseteq W$ (as W is g-invariant).

Claim (ii). $\overline{q}(w) = w$ for $w \in W$.

$$\overline{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(\underbrace{g^{-1}w}_{\in W}) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w = w.$$

So (i), (ii) $\Rightarrow$ $\overline{q}$ projects V onto W.

Claim (iii). If $h \in G$ then $h\overline{q}(v) = \overline{q}(hv)$ (for $v \in V$).

$$\begin{aligned} h\overline{q}(v) &= h\frac{1}{|G|}\sum_{g\in G} gq(g^{-1}v) = \frac{1}{|G|}\sum_{g\in G} hgq(g^{-1}v) = \frac{1}{|G|}\sum_{g\in G}(hg)q\big((hg)^{-1}hv\big) \\ &= \frac{1}{|G|}\sum_{g'\in G} g'q\big(g'^{-1}(hv)\big) = \overline{q}(hv). \end{aligned}$$

Claim (iv). $\ker\overline{q}$ is G-invariant.

If $v \in \ker\overline{q}$, $h \in G$, then $h\overline{q}(v) = 0 = \overline{q}(hv)$, so $hv \in \ker\overline{q}$.

Then $V = \operatorname{im}\overline{q} \oplus \ker\overline{q} = W \oplus \ker\overline{q}$ is a G-subspace decomposition. □

Remark. Complements are not necessarily unique.

The second proof uses inner products, hence we need to take $F = \mathbb{C}$ (or $\mathbb{R}$), and it can be generalised to compact groups (chapter 15).

Recall for V a $\mathbb{C}$-space, $\langle\,,\rangle$ is a **$\mathbb{C}$-inner product** if

(a) $\langle w, v\rangle = \overline{\langle v, w\rangle}$ for all v, w

(b) linear in LHS

(c) $\langle v, v\rangle > 0$ if $v \neq 0$

Additionally, $\langle\,,\rangle$ is **G-invariant** if

(d) $\langle gv, gw\rangle = \langle v, w\rangle$ for all $v, w \in V, g \in G$

Note that if W is a G-subspace of V (with G-invariant inner product) then $W^\perp$ is also G-invariant and $V = W \oplus W^\perp$.

Proof. Want: for all $v \in W^\perp$, for all $g \in G$, we have $gv \in W^\perp$.

Now, $v \in W^\perp \Leftrightarrow \langle v, w\rangle = 0$ for all $w \in W$. Thus $\langle gv, gw\rangle = 0$ for all $g \in G, w \in W$. Hence $\langle gv, w'\rangle = 0$ for all $w' \in W$ since we can take $w = g^{-1}w'$ by G-invariance of W. Hence $gv \in W^\perp$ since g was arbitrary. □

Hence if there is a G-invariant inner product on any complex G-space, we get:

(3.4′) (Weyl's Unitary Trick). Let ρ be a complex representation of the finite group G on the $\mathbb{C}$-space V. There is a G-invariant inner product on V (whence $\rho(G)$ is conjugate to a subgroup of $U(V)$, the **unitary** group on V, i.e. $\rho(g)^* = \rho(g^{-1})$).

Proof. There is an inner product on V: take basis $e_1, \ldots, e_n$, and define $(e_i, e_j) = \delta_{ij}$, extended sesquilinearly. Now define $\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}(gv, gw)$.

Claim. $\langle\,,\rangle$ is sesquilinear, positive definite, and G-invariant.

If $h \in G$, $\langle hv, hw\rangle = \frac{1}{|G|}\sum_{g\in G}\big((gh)v, (gh)w\big) = \frac{1}{|G|}\sum_{g'\in G}(g'v, g'w) = \langle v, w\rangle$. □

(3.5) (The (left) regular representation of G.) Define the **group algebra** of G to be the F-space $FG = \operatorname{span}\{e_g : g \in G\}$.

There is a G-linear action: for $h \in G$, define $h\big(\sum_g a_g e_g\big) = \sum_g a_g e_{hg} \big(= \sum_{g'} a_{h^{-1}g'} e_{g'}\big)$.

ρ_{reg} is the corresponding representation – the **regular representation** of G.

This is faithful of dimension $|G|$.

It turns out that *every* irreducible representation of G is a subrepresentation of ρ_{reg}.

(3.6) Proposition. Let ρ be an irreducible representation of the finite group G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg}.

Proof. Take $\rho : G \to GL(V)$, irreducible, and let $0 \neq v \in V$.

Let $\theta : FG \to V$, $\sum_g a_g e_g \mapsto \sum a_g \underset{\text{really } \rho(g)}{g} v$ (a G-homomorphism).

Now, V is irreducible and so $\operatorname{im}\theta = V$ (since $\operatorname{im}\theta$ is a G-subspace). Also $\ker\theta$ is a G-subspace of FG. Let W be a G-complement of $\ker\theta$ in FG (using (3.4)), so that $W < FG$ is a G-subspace and $FG = \ker\theta \oplus W$.

Hence $W \cong FG/\ker\theta \underset{G\text{-isom.}}{\cong} \operatorname{im}\theta = V$. □

More generally,

(3.7) Definition. Let F be a field, and let G act on a set X. Let $FX = \operatorname{span}\{e_x : x \in X\}$, with G-action $g\left(\sum_{x\in X} a_x e_x\right) = \sum a_x e_{gx}$.

So we have a G-space FX. The representation $G \to GL(V)$ with $V = FX$ is the corresponding **permutation representation**.

4. Schur's Lemma

(4.1) Theorem ('Schur's Lemma'). (a) Assume V, W are *irreducible* G-spaces (over a field F). Then any G-homomorphism $\theta : V \to W$ is either 0 or an isomorphism.

(b) Assume F is algebraically closed and let V be an irreducible G-space. Then any G-endomorphism $\theta : V \to V$ is a scalar multiple of the identity map id_V (a homothety).

Proof. (a) Let $\theta : V \to W$ be a G-homomorphism. Then $\ker\theta$ is a G-subspace of V, and since V is irreducible, either $\ker\theta = 0$ or $\ker\theta = V$. And $\mathrm{im}\,\theta$ is a G-subspace of W, so as W is irreducible, $\mathrm{im}\,\theta$ is either 0 or W. Hence either $\theta = 0$ or θ is injective and surjective, so θ is an isomorphism.

(b) Since F is algebraically closed, θ has an eigenvalue λ. Then $\theta - \lambda\mathrm{id}$ is a singular G-endomorphism on V, so must be 0, so $\theta = \lambda\mathrm{id}$. □

Recall from (2.8) the F-space $\mathrm{Hom}_G(V, W)$ of all G-homomorphisms $V \to W$. Write $\mathrm{End}_G(V)$ for the endomorphism algebra $\mathrm{Hom}_G(V, V)$.

(4.2) Corollary. If V, W are irreducible complex G-spaces, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are G-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V, W are not isomorphic then the only G-homomorphism $V \to W$ is 0 by (4.1). Assume $V \cong_G W$ and $\theta_1, \theta_2 \in \mathrm{Hom}_G(V, W)$, both $\neq 0$. Then θ_2 is invertible by (4.1) and $\theta_2^{-1}\theta_1 \in \mathrm{Hom}_G(V, V)$. So $\theta_2^{-1}\theta_1 = \lambda\mathrm{id}$ for some $\lambda \in \mathbb{C}$. Then $\theta_1 = \lambda\theta_2$. □

(4.3) Corollary. If G has a faithful complex irreducible representation then $Z(G)$ is cyclic.

Remark. The converse is false. (See examples sheet 1, question 11.)

Proof. Let $\rho : G \to GL(V)$ be a faithful irreducible complex representation. Let $z \in Z(G)$, so $zg = gz$ for all $g \in G$.

Consider the map $\phi_z : v \mapsto zv$ for $v \in V$. This is a G-endomorphism on V, hence is multiplication by a scalar μ_z, say (by Schur).

Then the map $Z(G) \to \mathbb{C}^\times$, $z \mapsto \mu_z$, is a representation of Z and is faithful (since ρ is). Thus $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$, hence is cyclic. □

Applications to abelian groups

(4.4) Corollary. The irreducible complex representations of a finite abelian group G are all 1-dimensional.

Proof. Either (1.4*) to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$ and if V is irreducible, then $V = \langle v \rangle$.

Or let V be an irreducible complex representation. For $g \in G$, the map $\theta_g : V \to V$, $v \mapsto gv$, is a G-endomorphism of V and, as V is irreducible, $\theta_g = \lambda_g\mathrm{id}$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any g. Thus, as V is irreducible, $V = \langle v \rangle$ is 1-dimensional. □

Remark. This fails on $\mathbb{R}$. E.g., C_3 has two irreducible real representations: one of dimension 1, one of dimension 2. (See sheet 1, question 12.)

Recall that any finite abelian group G is isomorphic to a product of cyclic groups, e.g. $C_6 \cong C_2 \times C_3$. In fact, it can be written as a product of C_{p^α} for various primes p and $\alpha \geqslant 1$, and the factors are uniquely determined up to ordering.

(4.5) Proposition. The finite abelian group $G = C_{n_1} \times \ldots \times C_{n_r}$ has precisely $|G|$ irreducible complex representations, as described below.

Proof. Write $G = \langle x_1 \rangle \times \ldots \times \langle x_r \rangle$ where $o(x_j) = n_j$. Suppose ρ is irreducible – so by (4.4) it's 1-dimensional, $\rho : G \to \mathbb{C}^\times$.

Let $\rho(1, \ldots, 1, x_j, 1, \ldots, 1) = \lambda_j \in \mathbb{C}^\times$. Then $\lambda_j^{n_j} = 1$, so λ_j is an n_j^{th} root of unity.

Now the values $(\lambda_1, \ldots, \lambda_r)$ determine ρ, as $\rho(x_1^{j_1}, \ldots, x_r^{j_r}) = \lambda_1^{j_1} \ldots \lambda_r^{j_r}$.

Thus $\rho \leftrightarrow (\lambda_1, \ldots, \lambda_r)$ with $\lambda_j^{n_j} = 1$ for all j. (And have $n_1 \ldots n_r$ such r-tuples, each giving a 1-dimensional representation.

Examples. (a) $G = C_4 = \langle x \rangle$.

	1	x	x^2	x^3
ρ_1	1	1	1	1
ρ_2	1	i	-1	i
ρ_3	1	-1	1	-1
ρ_4	1	$-i$	-1	i

(b) $G = V_4 = \langle x_1 \rangle \times \langle x_2 \rangle \cong C_2 \times C_2$.

	1	x_1	x_2	x_1x_2
ρ_1	1	1	1	1
ρ_2	1	1	-1	-1
ρ_3	1	-1	1	-1
ρ_4	1	-1	-1	1

Warning. There is no 'natural' 1-1 correspondence between the elements of G and the representations of G. If you choose an isomorphism $G \cong C_1 \times \ldots \times C_r$, then you can identify the two sets, but *it depends on the choice of isomorphism.*

**** Non-examinable section ****

Application to isotypical decompositions

(4.6) Proposition. Let V be a G-space over $\mathbb{C}$, and assume $V = U_1 \oplus \ldots \oplus U_n = W_1 \oplus \ldots \oplus W_n$, with all the U_j, W_k irreducible G-spaces. Let X be a fixed irreducible G-space. Let U be the sum of all the U_j isomorphic to X, and W be the sum of all the W_j isomorphic to X.

Then $U = W$, and is known as the **isotypical component** of V corresponding to X. Hence:

$\#\ U_j$ isomorphic to X = $\#\ W_k$ isomorphic to X = $(V : X)$ = **multiplicity** of X in V.

Proof (sketch). Look at $\theta_{jk} : U_j \xrightarrow{i_j} V \xrightarrow{\pi_k} W_k$, with $W_k \cong X$, where i_j is inclusion and π_k is projection. If $U_j \cong X$ then $U_j \subset W_k$ – all the projections to the other W_ℓ are 0.

'Then fiddle around with dimensions, then done.' $\square$

(4.7) Proposition. $(V : X) = \dim_{\mathbb{C}} \operatorname{Hom}_G(V, X)$ for X irreducible, V any G-space.

Proof. Prove $\operatorname{Hom}_G(W_1 \oplus W_2, X) \cong \operatorname{Hom}_G(W_1, X) \oplus \operatorname{Hom}_G(W_2, X)$, then apply Schur. (Or see James & Liebeck 11.6.)

(4.8) Proposition. $\dim_{\mathbb{C}} \operatorname{Hom}_G(\mathbb{C}G, X) = \dim_{\mathbb{C}} X$.

Proof. Let $d = \dim X$, and take a basis $\{e_1, \ldots, e_d\}$ of X. Define $\phi_i : \mathbb{C}G \to X$, $g \mapsto ge_i$ $(1 \leqslant i \leqslant d)$. Then $\phi_i \in \operatorname{Hom}_G(\mathbb{C}G, X)$ and $\{\phi_1, \ldots, \phi_d\}$ is a basis. (See James & Liebeck 11.8.)

Remark. If $V_1, \ldots, V_r$ are all the distinct complex irreducible G-spaces then $\mathbb{C}G = n_1V_1 \oplus \ldots \oplus n_rV_r$ where $n_i = \dim V_i$. Then $|G| = n_1^2 + \ldots + n_r^2$. (See (5.9), or James & Liebeck 11.2.)

Recall (2.17). $G = D_6$, $\mathbb{C}G = U_1 \oplus U_3 \oplus U_3 \oplus U_4$, $\dim \operatorname{Hom}(\mathbb{C}G, U_3) = 2$. (Challenge: find a basis for it.) U_1 and U_2 occur with multiplicity 1, and U_3 occurs with multiplicity 2.

**** End of non-examinable section ****

5. Character Theory

We want to attach invariants to a representation ρ of a finite group G on V. Matrix coefficients of $\rho(g)$ are basis dependent, so not true invariants.

Take $F = \mathbb{C}$, and G finite. $\rho = \rho_V : G \to GL(V)$, a representation of G.

(5.1) Definition. The **character** $\chi_\rho = \chi_V = \chi$ is defined by $\chi(g) = \operatorname{tr}\rho(g)$ $(= \operatorname{tr} R(g)$, where $R(g)$ is any matrix representation of $\rho(g)$ with respect to any basis). The **degree** of χ_V is $\dim V$.

Thus χ is a function $G \to \mathbb{C}$. χ is linear if $\dim V = 1$, in which case χ is a homomorphism $G \to \mathbb{C}^\times$.

- χ is **irreducible** if ρ is.
- χ is **faithful** if ρ is.
- χ is **trivial** (**principal**) if ρ is the trivial representation: write $\chi = 1_G$.

χ is a **complete invariant** in the sense that it determines ρ up to isomorphism – see (5.7).

(5.2) First properties.

(i) $\chi_V(1) = \dim V$.

(ii) χ_V is a **class function**, viz it is conjugation invariant, i.e. $\chi_V(hgh^{-1}) = \chi_V(g)$ for all $g, h \in G$.
Thus χ_V is constant on the conjugacy classes (ccls) of G.

(iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

(iv) For two representations, V, W, have $\chi_{V\oplus W} = \chi_V + \chi_W$.

Proof. (i) $\operatorname{tr}(I) = n$.

(ii) $\chi(hgh^{-1}) = \operatorname{tr}(R_h R_g R_{h^{-1}}) = \operatorname{tr}(R_g) = \chi(g)$.

(iii) $g \in G$ has finite order, so by (1.4) can assume $\rho(g)$ is represented by a diagonal matrix $\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$. Thus $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by $\begin{pmatrix}\lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1}\end{pmatrix}$, and since $|\lambda_i| = 1$ for all i, $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi(g)}$.

(iv) Suppose $V = V_1 \oplus V_2$, $\rho_i : G \to GL(V_i)$, $\rho : G \to GL(V)$. Take basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ of V, containing bases $\mathcal{B}_i$ of V_i.
With respect to $\mathcal{B}$, $\rho(g)$ has matrix $\begin{bmatrix}[\rho_1(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_2(g)]_{\mathcal{B}_2}\end{bmatrix}$.
So $\chi(g) = \operatorname{tr}(\text{this}) = \operatorname{tr}\rho_1(g) + \operatorname{tr}\rho_2(g) = \chi_1(g) + \chi_2(g)$. □

Remark. We see later that χ_1, χ_2 characters of $G \Rightarrow \chi_1\chi_2$ also a character of G. This uses tensor products – see (9.6).

(5.3) Lemma. Let $\rho : G \to GL(V)$ be a complex representation **affording** ("which can take") the character χ. Then $|\chi(g)| \leqslant \chi(1)$, with equality iff $\rho(g) = \lambda\mathrm{id}$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1) \Leftrightarrow g \in \ker\rho$.

Proof. Fix g. W.r.t. a basis of V of eigenvectors of $\rho(g)$, the matrix of $\rho(g)$ is $\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$.

Hence, $|\chi(g)| = |\sum \lambda_i| \leqslant \sum |\lambda_i| = \sum 1 = \dim V = \chi(1)$, with equality iff all λ_j are equal to λ, say (using Cauchy-Schwarz). And if $\chi(g) = \chi(1)$ then $\rho(g) = \lambda\mathrm{id}$.

Therefore, $\chi(g) = \lambda\chi(1)$, and so $\lambda = 1$ and $g \in \ker\rho$. $\square$

(5.4) Lemma. If χ is a complex irreducible character of G, then so is $\overline{\chi}$, and so is $\varepsilon\chi$ for any linear character ε of G.

Proof. If $R : G \to GL_n(\mathbb{C})$ is a complex (matrix) representation then so is $\overline{R} : G \to GL_n(\mathbb{C})$, $g \mapsto \overline{R(g)}$.

Similarly for $R' : g \mapsto \varepsilon(g)R(g)$. Check the details. $\square$

(5.5) Definition. $\mathcal{C}(G) = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g)\ \forall\, h, g \in G\}$, the $\mathbb{C}$-space of class functions. (Where $f_1 + f_2 : g \mapsto f_1(g) + f_2(g)$, $\lambda f : g \mapsto \lambda f(g)$.)

List conjugacy classes as $\mathcal{C}_1(= \{1\})$, $\mathcal{C}_2$, …, $\mathcal{C}_k$. Choose $g_1(= 1)$, g_2, …, g_k as representatives of the classes.

Note also that $\dim_{\mathbb{C}} \mathcal{C}(G) = k$, as the characteristic functions δ_j of the conjugacy classes form a basis, where $\delta_j(g) = 1$ if $g \in \mathcal{C}_j$, and 0 otherwise.

Define Hermitian inner product on $\mathcal{C}(G)$ by

$$\langle f, f' \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} f'(g) = \frac{1}{|G|} \sum_{j=1}^{k} |\mathcal{C}_j|\, \overline{f(g_j)} f'(g_j) = \sum_{j=1}^{k} \frac{1}{|C_G(g_j)|}\, \overline{f(g_j)} f'(g_j)$$

using orbit-stabiliser: $|\mathcal{C}(x)| = |G : C_G(x)|$, where $C_G(x)$ is the centraliser of x in G.

For characters, $\langle \chi, \chi' \rangle = \displaystyle\sum_{j=1}^{k} \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$ is a real symmetric form.

Main result follows.

(5.6) Big Theorem (Completeness of characters). The $\mathbb{C}$-irreducible characters of G form an orthonormal basis of the space of class functions of G. Moreover,

(a) If $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ are irreducible representations of G affording characters χ, χ' then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \rho, \rho' \text{ are isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

(b) Each class function of G can be expressed as a linear combinations of irreducible characters of G.

Proof. In chapter 6.

(5.7) Corollary. Complex representations of finite groups are characterised by their characters.

Proof. Have $\rho : G \to GL(V)$ affording χ. Complete reducibility (3.3) says $\rho = m_1\rho_1 \oplus \ldots \oplus m_k\rho_k$, where ρ_j is irreducible and $m_j \geqslant 0$. Then $m_j = \langle \chi, \chi_j \rangle$ where χ_j is afforded by ρ_j, since $\chi = m_1\chi_1 + \ldots + m_k\chi_k$ and $\langle \chi, \chi_j \rangle = \langle m_1\chi_1 + \ldots + m_k\chi_k, \chi_j \rangle = m_j$, by (5.6)(a). $\square$

(5.8) Corollary (Irreducibility criterion). If ρ is a complex representation of G affording χ then ρ irreducible $\Leftrightarrow \langle \chi, \chi \rangle = 1$.

Proof. ($\Rightarrow$) Orthonormality.

($\Leftarrow$) Assume $\langle \chi, \chi \rangle = 1$. (3.3) says $\chi = \sum m_j \chi_j$, for χ_j irreducible, $m_j \geqslant 0$. Then $\sum m_j^2 = 1$, so $\chi = \chi_j$ for some j. Therefore χ is irreducible.

(5.9) Theorem. If the irreducible complex representations of G have dimensions $n_1, \ldots, n_k$, then $|G| = \sum_i n_i^2$.

(Recall end of chapter 4.)

Proof. Recall from (3.5), $\rho_{\text{reg}} : G \to GL(\mathbb{C}G)$, the regular representation of G, of dimension $|G|$. Let π_{reg} be its character.

Claim. $\pi_{\text{reg}}(1) = |G|$ and $\pi_{\text{reg}}(h) = 0$ if $h \neq 1$.

Proof. Easy. Let $G = \{g_1, \ldots, g_n\}$ and take $h \in G$, $h \neq 1$. For $1 \leqslant i \leqslant n$, $hg_i = g_j$, some $j \neq i$, so i^{th} row of $[\rho_{\text{reg}}(h)]_{\mathcal{B}}$ has 0s in every place, except column j – in particular, the $(i,i)^{\text{th}}$ entry is 0 for all i. Hence $\pi_{\text{reg}}(h) = \operatorname{tr}[\rho_{\text{reg}}(h)]_{\mathcal{B}} = 0$.

By claim, $\pi_{\text{reg}} = \sum n_j \chi_j$ with $n_j = \chi_j(1)$:

$$n_j = \langle \pi_{\text{reg}}, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\pi_{\text{reg}}(g)} \chi_j(g) = \frac{1}{|G|} |G| \, \chi_j(1) = \chi_j(1)$$

□

(5.10) Corollary. The number of irreducible characters of G (up to equivalence) equals k, the number of conjugacy classes.

(5.11) Corollary. Elements $g_1, g_2 \in G$ are conjugate iff $\chi(g_1) = \chi(g_2)$ for all irreducible characters of G.

Proof. ($\Rightarrow$) Characters are class functions.

($\Leftarrow$) Let δ be the characteristic function of the class of g_1. Then δ is a class function, so can be written as a linear combination of the irreducible characters of G, by (5.6)(b). Hence $\delta(g_2) = \delta(g_1) = 1$. So $g_2 \in \mathcal{C}_G(g_1)$. □

Recall from (5.5) the inner product on $\mathcal{C}(G)$ and the real symmetric form $\langle \, , \rangle$ for characters.

(5.12) Definition. G finite, $F = \mathbb{C}$. The **character table** of G is the $k \times k$ matrix $X = [\chi_i(g_j)]$ where $\chi_1 (= 1)$, χ_2, …, χ_k are the irreducible characters of G, and $\mathcal{C}_1 (= \{1\})$, $\mathcal{C}_2$, …, $\mathcal{C}_k$ are the conjugacy classes, with $g_j \in \mathcal{C}_j$.

I.e., the $(i,j)^{\text{th}}$ entry of X is $\chi_i(g_j)$.

Examples. $C_2 = \langle x : x^2 = 1 \rangle$

	1	x
χ_1	1	1
χ_2	1	-1

$C_3 = \langle x : x^3 = 1 \rangle$

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = e^{2\pi i/3} \in \mu_3$.

$G = D_6 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1}\rangle \cong S_3$.

In (2.17) we found a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules: U_1, U_2, U_3. Let $\chi_i = \chi_{U_i}$, $(1 \leqslant i \leqslant 3)$.

	1	$\{a, a^2\}$	$\{b, ab, a^2b\}$	$\leftarrow g_j$
χ_1	1	1	1	
χ_2	1	1	-1	
χ_3	2	-1	0	
	$\nwarrow 1^2+1^2+2^2=6$			
	6	3	2	$\leftarrow \lvert C_G(g_j)\rvert$
	1	2	3	$\leftarrow \lvert \mathcal{C}(g_j)\rvert$

Orthogonality:

$$\frac{1\times 2}{6} + \frac{(-1)(1)}{3} = 0 \quad \text{(rows 2 \& 3)}$$

$$\frac{2^2}{6} + \frac{(-1)^2}{3} = 1 \quad \text{(row 3)}$$

6. Proofs and Orthogonality

We want to prove (5.6), the Big Theorem. We'll do this in two ways.

Proof 1 of (5.6)(a). Fix bases of V and V'. Write $R(g)$, $R'(g)$ for the matrices of $\rho(g)$, $\rho'(g)$ with respect to these, respectively.

$$\langle \chi', \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi'(g^{-1}) \chi(g) = \frac{1}{|G|} \sum_{\substack{g \in G \\ 1 \leqslant i \leqslant n' \\ 1 \leqslant j \leqslant n}} R'(g^{-1})_{ii} R(g)_{jj}$$

Let $\phi : V \to V'$ be linear, and define $\widetilde{\phi} : V \to V'$, $v \mapsto \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \rho'(g^{-1}) \phi \rho(g)(v)$.

Then this 'average' $\widetilde{\phi}$ is a G-homomorphism. For if $h \in G$,

$$\rho'(h^{-1}) \widetilde{\phi} \rho(h)(v) = \frac{1}{|G|} \sum_{g \in G} \rho'\big((gh)^{-1}\big) \phi \big(\rho(gh)\big)(v) = \frac{1}{|G|} \sum_{g' \in G} \rho'(g'^{-1}) \phi \rho(g')(v) = \widetilde{\phi}(v).$$

Assume first that ρ, ρ' are *not* isomorphic. Schur's Lemma says $\widetilde{\phi} = 0$ for any linear $\phi : V \to V'$.

Let $\phi = \varepsilon_{\alpha\beta}$ having matrix $E_{\alpha\beta}$ (with respect to our basis), namely 0 everywhere except 1 in the $(\alpha, \beta)^{\text{th}}$ place.

Then $\widetilde{\varepsilon}_{\alpha\beta} = 0$, so $\dfrac{1}{|G|} \displaystyle\sum_{g \in G} \big(R'(g^{-1}) E_{\alpha\beta} R(g)\big)_{ij} = 0$.

Thus $\dfrac{1}{|G|} \displaystyle\sum_{g \in G} R'(g^{-1})_{i\alpha} R(g)_{\beta j} = 0$ for all i, j.

With $\alpha = i, \beta = j$, $\dfrac{1}{|G|} \displaystyle\sum_{g \in G} R'(g^{-1})_{ii} R(g)_{jj} = 0$. Sum over i, j and conclude $\langle \chi', \chi \rangle = 0$.

Now assume that ρ, ρ' are isomorphic, so $\chi = \chi'$. Take $V = V'$, $\rho = \rho'$. If $\phi : V \to V$ is linear, then $\widetilde{\phi} \in \mathrm{Hom}_G(V, V)$.

Now $\operatorname{tr} \phi = \operatorname{tr} \widetilde{\phi}$, as $\operatorname{tr} \widetilde{\phi} = \dfrac{1}{|G|} \displaystyle\sum \operatorname{tr}\big(\rho(g^{-1}) \phi \rho(g)\big) = \dfrac{1}{|G|} \displaystyle\sum \operatorname{tr} \phi = \operatorname{tr} \phi$.

By Schur, $\widetilde{\phi} = \lambda \mathrm{id}$ for some $\lambda \in \mathbb{C}$ (depending on ϕ). Now $\lambda = \dfrac{1}{n} \operatorname{tr} \phi$.

Let $\phi = \varepsilon_{\alpha\beta}$, so $\operatorname{tr} \phi = \delta_{\alpha\beta}$. Hence $\widetilde{\varepsilon}_{\alpha\beta} = \dfrac{1}{n} \delta_{\alpha\beta} \mathrm{id} = \dfrac{1}{|G|} \displaystyle\sum_g \rho(g^{-1}) \varepsilon_{\alpha\beta} \rho(g)$.

In terms of matrices, take the $(i, j)^{\text{th}}$ entry: $\dfrac{1}{|G|} \displaystyle\sum_g R(g^{-1})_{i\alpha} R(g)_{\beta j} = \dfrac{1}{n} \delta_{\alpha\beta} \delta_{ij}$,

and put $\alpha = i$, $\beta = j$ to get $\dfrac{1}{|G|} \displaystyle\sum_g R(g^{-1})_{ii} R(g)_{jj} = \dfrac{1}{n} \delta_{ij}$.

Finally sum over i, j: $\langle \chi, \chi \rangle = 1$. $\square$

Before proving (b), let's prove column orthogonality, assuming (5.10).

(6.1) Theorem (column orthogonality). $\displaystyle\sum_{i=1}^{k} \overline{\chi_i(g_j)}\chi_i(g_\ell) = \delta_{jl}\,|C_G(g_j)|$.

This has an easy corollary:

(6.2) Corollary. $|G| = \displaystyle\sum_{i=1}^{k} \chi_i^2(1)$.

Proof of (6.1). $\delta_{ij} = \langle \chi_i, \chi_j \rangle = \displaystyle\sum_\ell \frac{1}{|C_G(g_\ell)|}\,\overline{\chi_i(g_\ell)}\chi_j(g_\ell)$.

Consider the character table $X = (\chi_i(g_j))$.

Then $\overline{X}D^{-1}X^t = I_{k\times k}$, where $D = \begin{pmatrix} |C_G(g_1)| & & \\ & \ddots & \\ & & |C_G(g_k)| \end{pmatrix}$.

As X is a square matrix, it follows that $D^{-1}\overline{X}^t$ is the inverse of X. So $\overline{X}^t X = D$. □

Proof of (5.6)(b). List all the irreducible characters $\chi_1, \ldots, \chi_\ell$ of G. It's enough to show that the orthogonal complement of span$\{\chi_1, \ldots, \chi_\ell\}$ in $\mathcal{C}(G)$ is 0.

To see this, assume $f \in \mathcal{C}(G)$ with $\langle f, \chi_j \rangle = 0$ for all irreducible χ_j.

Let $\rho : G \to GL(V)$ be irreducible affording $\chi \in \{\chi_1, \ldots, \chi_\ell\}$. Then $\langle f, \chi \rangle = 0$.

Consider $\displaystyle\frac{1}{|G|}\sum \overline{f(g)}\rho(g) : V \to V$. This is a G-homomorphism, so as ρ is irreducible it must be λid for some $\lambda \in \mathbb{C}$ (by Schur).

Now, $n\lambda = \operatorname{tr} \displaystyle\frac{1}{|G|}\sum \overline{f(g)}\rho(g) = \frac{1}{|G|}\sum \overline{f(g)}\chi(g) = 0 = \langle f, \chi \rangle$.

So $\lambda = 0$. Hence $\sum \overline{f(g)}\rho(g) = 0$, the zero endomorphism on V, for *all* representations ρ. Take $\rho = \rho_{\text{reg}}$, where $\rho_{\text{reg}}(g) : e_1 \mapsto e_g$ $(g \in G)$, the regular representation.

So $\sum_g \overline{f(g)}\rho_{\text{reg}}(g) : e_1 \mapsto \sum_g \overline{f(g)}e_g$. It follows that $\sum \overline{f(g)}e_g = 0$.

Therefore $\overline{f(g)} = 0$ for all $g \in G$. And so $f = 0$. □

Various important corollaries follow from this:

- (5.10) # irreducibles of G = # conjugacy classes
- (6.1) column orthogonality
- (6.2) $|G| = \sum \chi_i^2(1) = \sum_{i=1}^k n_i^2$
- (5.11) irreducible χ is constant on conjugacy classes – i.e., $g_1 \sim_G g_2 \Rightarrow \chi(g_1) = \chi(g_2)$.
- g, g^{-1} are G-conjugate $\Leftrightarrow \chi(g) \in \mathbb{R}$ for all irreducible χ – as $\chi(g^{-1}) = \overline{\chi(g)}$.

Example.

	6	3	2	$\leftarrow \lvert C_G(g_j)\rvert$
	1	a	b	$\leftarrow g_j$
χ_1	1	1	1	
χ_2	1	1	-1	
χ_3	2	-1	0	$\leftarrow$ coming from operations on equilateral triangle

Column orthogonality: $\displaystyle\sum_{i=1}^{3} \overline{\chi_i(g_r)}\chi_i(g_s)$.

$$\begin{array}{llcll} r=1, s=2: & 1.1+1.1+2(-1) & = & 0 & r \neq s \\ r=1, s=3: & 1.1+1(-1)+2.0 & = & 0 & r \neq s \\ r=2, s=2: & 1.1+1.1+(-1)(-1) & = & 3 & r = s, \text{ weight by } |C_G(g_r)| \end{array}$$

**** Non-examinable section ****

Proof 2 of (5.6)(a). (Uses starred material at the end of chapter 4.)

X irreducible G-space, V any G-space. $V = \bigoplus_{i=1}^m U_i$, with U_i irreducible.

Then the number of U_j isomorphic to X is independent of the decomposition. We wrote $(V : X)$ for this number, and in (4.7) we saw $(V : X) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, X)$ $(*)$.

Let $\rho : G \to GL(U)$ have character χ Write $U^G = \{u \in U : \rho(g)u = u \,\forall\, g \in G\}$, the **$G$-invariants** of U.

Consider the map $\pi : U \to U$, $u \mapsto \dfrac{1}{|G|}\displaystyle\sum_g \rho(g)u$.

This is a projection onto U^G (because it's a G-homomorphism, and when restricted to U^G it acts as the identity there). Verify $\dim U^G = \operatorname{tr}\pi = \frac{1}{|G|}\sum_g \chi_U(g)$ $(**)$ (by decomposing U and looking at bases).

Now choose $U = \mathrm{Hom}_{\mathbb{C}}(V, V')$ with V, V' being G-spaces. G acts on U via $g.\theta(v) = \rho_V(g)\big(\theta\rho_{V'}(g^{-1})v\big)$ for $\theta \in U$.

But $\mathrm{Hom}_G(V, V') = \big(\mathrm{Hom}_{\mathbb{C}}(V, V')\big)^G$, so by $(**)$, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V') = \dfrac{1}{|G|}\displaystyle\sum_g \chi_U(g)$.

Finally, show $\chi_U(g) = \chi_{V'}(g^{-1})\chi_V(g)$ – see section on tensor products in chapter 9.

The orthogonality of the irreducible characters now follows from $(*)$. $\square$

**** End of non-examinable section ****

7. Permutation Representations

Preview was given in (3.7). Recall:

- G finite, acting on finite set $X = \{x_1, \ldots, x_n\}$.
- $\mathbb{C}X = \mathbb{C}$-space, basis $\{e_{x_1}, \ldots, e_{x_n}\}$ of dimension $|X|$. $\mathbb{C}X = \left\{\sum_j a_j e_{x_j} : a_j \in \mathbb{C}\right\}$.
- corresponding permutation representation, $\rho_X : G \to GL(\mathbb{C}X)$, $g \mapsto \rho(g)$, where $\rho(g) : e_{x_j} \mapsto e_{gx_j}$, extended linearly. So $\rho_X(g) : \sum_{x \in X} a_x e_x \mapsto \sum_{x \in X} a_x e_{gx}$.
- ρ_X is the **permutation representation** corresponding to the action of G on X.
- matrices representing $\rho_X(g)$ with respect to the basis $\{e_x\}_{x \in X}$ are permutation matrices: 0 everywhere except one 1 in each row and column, and $(\rho(g))_{ij} = 1$ precisely when $gx_j = x_i$.

(7.1) Permutation character π_X is $\pi_X(g) = |\mathrm{fix}_X(g)| = |\{x \in X : gx = x\}|$.

(7.2) π_X always contains 1_G. For: span$(e_{x_1} + \ldots + e_{x_n})$ is a trivial G-subspace of $\mathbb{C}X$ with G-invariant complement span$(\sum a_x e_x : \sum a_x = 0)$.

(7.3) 'Burnside's Lemma' (Cauchy, Frobenius). $\langle \pi_X, 1 \rangle = \#$ orbits of G on X.

Proof. If $X = X_1 \cup \ldots \cup X_\ell$, a disjoint union of orbits, then $\pi_X = \pi_{X_1} + \ldots + \pi_{X_\ell}$ with π_{X_j} the permutation character of G on X_j. So to prove the claim, it's enough to show that if G is transitive on X then $\langle \pi_X, 1 \rangle = 1$.

So, assume G is transitive on X. Then

$$\begin{aligned} \langle \pi_X, 1 \rangle &= \frac{1}{|G|} \sum_{g \in G} \pi_X(g) \\ &= \frac{1}{|G|} \left| \left\{ (g, x) \in G \times X : gx = x \right\} \right| \\ &= \frac{1}{|G|} \sum_{x \in X} |G_x| \qquad (G_x = \text{ stabiliser of } x) \\ &= \frac{1}{|G|} |X| \, |G_x| \;=\; \frac{1}{|G|} |G| \;=\; 1 \end{aligned}$$

□

(7.4) Lemma. Let G act on sets X_1, X_2. Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1 \times X_2} = \pi_{X_1} \pi_{X_2}$ and so $\langle \pi_{X_1}, \pi_{X_2} \rangle = \#$ orbits of G on $X_1 \times X_2$.

Proof. $\langle \pi_{X_1}, \pi_{X_2} \rangle = \langle \pi_{X_1} \pi_{X_2}, 1 \rangle = \langle \pi_{X_1 \times X_2}, 1 \rangle = \#$ orbits of G on $X_1 \times X_2$ (by (7.3)). □

(7.5) Let G act on X, $|X| > 2$. Then G is **2-transitive** on X if G has just two orbits on $X \times X$, namely $\{(x, x) : x \in X\}$ and $\{(x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

(7.6) Lemma. Let G act on X, $|X| > 2$. Then $\pi_X = 1 + \chi$ with χ irreducible $\Leftrightarrow$ G is 2-transitive on X.

Proof. $\pi_X = m_1 1 + m_2 \chi_2 + \ldots + m_\ell \chi_\ell$ with $1, \chi_2, \ldots, \chi_\ell$ distinct irreducibles and $m_i \in \mathbb{Z}_{\geqslant 0}$. Then $\langle \pi_X, \pi_X \rangle = \sum_{i=1}^{l} m_i^2$. Hence G is 2-transitive on X iff $\ell = 2$, $m_1 = m_2 = 1$. □

(7.7) S_n acting on X_n (see 1.6) is 2-transitive. Hence $\pi_{X_n} = 1 + \chi$ with χ irreducible of degree $n - 1$. Similarly for A_n $(n > 3)$

(7.8) Example. $G = S_4$.

Conjugacy classes correspond to different cycle types.

		1	3	8	6	6	$\leftarrow$ sizes
		1	(12)(34)	(123)	(1234)	(12)	$\leftarrow$ ccl reps
trivial $\to$	χ_1	1	1	1	1	1	two linear characters since $S_4/S_4' = C_2$
sign $\to$	χ_2	1	1	1	-1	-1	
$\pi_{X_4} - 1 \to$	χ_3	3	-1	0	-1	1	
$\chi_3 \times \chi_2 \to$	χ_4	3	-1	0	1	-1	$\leftarrow$ product of the two above
	χ_5	d	x	y	z	w	

Know: $24 = 1 + 1 + 9 + 9 + d^2 \;\Rightarrow\; d = 2$.

Column orthogonality:

$$\begin{aligned} 1+1-3-3+2x &= 0 &\Rightarrow\quad & x = 2 \\ 1+1+0+0+2y &= 0 &\Rightarrow\quad & y = -1 \\ 1-1-3+3+2z &= 0 &\Rightarrow\quad & z = 0 \\ 1-1+3-3+2w &= 0 &\Rightarrow\quad & w = 0 \end{aligned}$$

Or: $\chi_{\text{reg}} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5 \;\Rightarrow\; \chi_5 = \frac{1}{2}(\chi_{\text{reg}} - \chi_1 - \chi_2 - 3\chi_3 - 3\chi_4)$.

Or: can obtain χ_5 by observing $S_4/V_4 \cong S_3$ and 'lifting' characters – see chapter 8.

(7.9) Example. $G = S_5$.

		1	15	20	24	10	20	30	$\leftarrow\lvert\mathcal{C}_j\rvert$
		1	(12)(34)	(123)	(12345)	(12)	(123)(45)	(1234)	$\leftarrow g_j$
trivial $\to$	χ_1	1	1	1	1	1	1	1	
sign $\to$	χ_2	1	1	1	1	-1	-1	-1	
$\pi_{X_5} - 1 \to$	χ_3	4	0	1	-1	2	-1	0	
$\chi_3 \times \chi_2 \to$	χ_4	4	0	1	-1	-2	1	0	
	χ_5	5	1	-1	0	-1	-1	1	
$\chi_5 \times \chi_2 \to$	χ_6	5	1	-1	0	1	1	-1	
	χ_7	6	-2	0	1	0	0	0	

There are various methods to get χ_5, χ_6 of degree 5.

One way is to note that if $X = \mathrm{Syl}_5(G)$ then $|X| = 6$ and one checks that $\langle \chi_X, \chi_X \rangle = 2$. Therefore $\pi_X - 1$ is irreducible.

For χ_7, first $\sum d_i^2 = 120$ gives $\deg \chi_7 = 6$, and orthogonality for the remaining entries.

Or: let S_5 act on the set of $\binom{5}{2}$ unordered pairs of elements of $\{1, 2, 3, 4, 5\}$.

$\pi_{\binom{5}{2}} : 10 \quad 2 \quad 1 \quad 0 \quad 4 \quad 1 \quad 0$

$$\left.\begin{array}{l} \langle \chi_{\binom{5}{2}}, \chi_{\binom{5}{2}} \rangle = 3 \\ \langle \chi_{\binom{5}{2}}, 1 \rangle = 1 \\ \langle \chi_{\binom{5}{2}}, \chi_3 \rangle = 1 \end{array}\right\} \Rightarrow \chi_{\binom{5}{2}} = 1 + \chi_3 + \psi$$

ψ has degree 5 (and is actually χ_6 in the table).

See chapter 10 for the method of induced characters, and chapter 9 for symmetric and alternating powers.

(7.10) Alternating groups.

Let $g \in A_n$. Then
$$\begin{array}{rcl} |\mathcal{C}_{S_n}(g)| & = & |S_n : C_{S_n}(g)| \\ \cup\!\mid & & \uparrow A_n \text{ index 2 in } S_n \\ |\mathcal{C}_{A_n}(g)| & = & |A_n : C_{A_n}(g)| \end{array}$$

but not necessarily equal: e.g., if $\sigma = (123)$, then $\mathcal{C}_{A_n}(\sigma) = \{\sigma\}$, but $\mathcal{C}_{S_n}(\sigma) = \{\sigma, \sigma^{-1}\}$.

We know $|S_n : A_n| = 2$, and in fact:

(7.11) If $g \in A_n$ then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$ precisely when g commutes with some odd permutation; otherwise it breaks up into two classes of equal size. (In the latter case, precisely when the disjoint cycle decomposition of g is a product of odd cycles of distinct lengths.)

Proof. See James & Liebeck 12.17.

(7.12) $G = A_4$. Write $\omega = e^{2\pi i/3}$.

$$\begin{array}{rc|cccc l} & & 1 & 3 & 4 & 4 & \leftarrow |\mathcal{C}_j| \\ & & 1 & (12)(34) & (123) & (123)^{-1} & \leftarrow g_j \\ \hline 1_G \rightarrow & \chi_1 & 1 & 1 & 1 & 1 & \\ \pi_X - 1 \rightarrow & \chi_2 & 3 & -1 & 0 & 0 & \\ & \chi_3 & 1 & 1 & \omega & \omega^2 & \\ & \chi_4 & 1 & 1 & \omega^2 & \omega & \\ & & \uparrow & & & & \\ & & \multicolumn{5}{l}{\sum d_i^2 = 12 = 1^2 + 3^2 + ?^2 + ?^2 \Rightarrow ? = 1} \end{array}$$

Final two linear characters are found via $G/G' = G/V_4 = C_3$, by lifting – see chapter 8.

For A_5 see Telemann chapter 11, or James & Liebeck 20.13.

8. Normal Subgroups and Lifting Characters

(8.1) Lemma. Let $N \lhd G$, let $\widetilde{\rho} : G/N \to GL(V)$ be a representation of G/N. Then $\rho : G \xrightarrow{q} G/N \xrightarrow{\widetilde{\rho}} GL(V)$ is a representation of G, where $\rho(g) = \widetilde{\rho}(gN)$ (and q is the natural homomorphism). Moreover, ρ is irreducible if $\widetilde{\rho}$ is.

The corresponding characters satisfy $\chi(g) = \widetilde{\chi}(gN)$ for $g \in G$, and $\deg \chi = \deg \widetilde{\chi}$. We say that $\widetilde{\chi}$ **lifts** to χ.

The lifting sending $\widetilde{\chi} \mapsto \chi$ is a bijection between

$$\{\text{irreducibles of } G/N\} \longleftrightarrow \{\text{irreducibles of } G \text{ with } N \text{ in the kernel}\}$$

Proof. (See examples sheet 1, question 4.)

Note: $\chi(g) = \operatorname{tr}(\rho(g)) = \operatorname{tr}(\widetilde{\rho}(gN)) = \widetilde{\chi}(gN)\ \forall\, g$, and $\chi(1) = \widetilde{\chi}(N)$, so $\deg \chi = \deg \widetilde{\chi}$.

Bijection. If $\widetilde{\chi}$ is a character of G/N and χ is a lift to G then $\widetilde{\chi}(N) = \chi(1)$. Also, if $k \in N$ then $\chi(k) = \widetilde{\chi}(kN) = \widetilde{\chi}(N) = \chi(1)$. So $N \leqslant \ker \chi$.

Now let χ be a character of G with $N \leqslant \ker \chi$. Suppose $\rho : G \to GL(V)$ affords χ. Define $\widetilde{\rho} : G/N \to GL(V)$, $gN \mapsto \rho(g)$ for $g \in G$. This is well-defined (as $N \leqslant \ker \chi$) and $\widetilde{\rho}$ is a homomorphism, hence a representation of G/N. If $\widetilde{\chi}$ is the character of $\widetilde{\rho}$ then $\widetilde{\chi}(gN) = \chi(g)$ for all $g \in G$.

Finally, check irreducibility is preserved. $\square$

Definition. The **derived subgroup** of G is $G' = \langle [a,b] : a, b \in G \rangle$, where $[a,b] = aba^{-1}b^{-1}$ is the **commutator** of a and b. (G' is a crude measure of how abelian G is.)

(8.2) Lemma. G' is the unique minimal normal subgroup of G such that G/G' is abelian. (I.e., G/N abelian $\Rightarrow G' \leqslant N$, and G/G' is abelian.)

G has precisely $\ell = |G/G'|$ representations of degree 1, all with kernel containing G' and obtained by lifting from G/G'.

Proof. $G' \lhd G$ – easy exercise.

Let $N \lhd G$. Let $g, h \in G$. Then $g^{-1}h^{-1}gh \in N \Leftrightarrow ghN = hgN \Leftrightarrow (gN)(hN) = (hN)(gN)$. So $G' \leqslant N \Leftrightarrow G/N$ abelian. Since $G' \lhd G$, G/G' is an abelian group.

By (4.5), G/G' has exactly ℓ irreducible characters, $\chi_1, \ldots, \chi_\ell$, all of degree 1. The lifts of these to G also have degree 1 and by (8.1) these are precisely the irreducible characters χ_i of G such that $G' \leqslant \ker \chi_i$.

But any linear character χ of G is a homomorphism $\chi : G \to \mathbb{C}^\times$, hence $\chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g^{-1})\chi(h^{-1}) = 1$.

Therefore $G' \leqslant \ker \chi$, and so $\chi_1, \ldots, \chi_\ell$ are all irreducible characters of G. $\square$

Examples. (i) Let $G = S_n$. Show $G' = A_n$. Thus $G/G' \cong C_2$.

So S_n must have exactly two linear characters.

(ii) $G = A_4$.

	1	$(12)(34)$	(123)	$(123)^2$
1_G	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

($\omega = e^{2\pi i/3}$)

Let $N = \{1, (12)(34), (13)(24), (14)(23)\} \leqslant G$. In fact, $N \cong V_4$, $N \lhd G$, and $G/N \cong C_3$.

Also, $G' = V_4$, so $G/G' \cong C_3$.

(8.3) Lemma. G is *not* simple iff $\chi(g) = \chi(1)$ for some irreducible character $\chi \neq 1_G$ and $1 \neq g \in G$. Any normal subgroup of G is the intersection of kernels of some of the irreducibles of G, $N = \bigcap\limits_{\chi_i \text{ irred}} \ker \chi_i$.

Proof. If $\chi(g) = \chi(1)$ for some non-principal character χ (afforded by ρ, say), then $g \in \ker \rho$ (by (5.3)). Therefore if $g \neq 1$ then $1 \neq \ker \rho \lhd G$.

If $1 \neq N \lhd G$, take an irreducible $\widetilde{\chi}$ of G/N ($\widetilde{\chi} \neq 1_{G/N}$). Lift to get an irreducible χ afforded by ρ of G, then $N \leqslant \ker \rho \lhd G$. Therefore $\chi(g) = \chi(1)$ for $g \in N$.

In fact, if $1 \neq N \lhd G$ then N is the intersection of the kernels of the lifts of all of the irreducibles of G/N. $\leqslant$ is clear. For $\geqslant$: if $g \in G \setminus N$ then $gN \neq N$, so $\widetilde{\chi}(gN) \neq \widetilde{\chi}(N)$ for some irreducible $\widetilde{\chi}$ of G/N, and then lifting $\widetilde{\chi}$ to χ we have $\chi(g) \neq \chi(1)$. $\square$

9. Dual Spaces and Tensor Products of Representations

Recall (5.5), (5.6): $\mathcal{C}(G) = \mathbb{C}$-space of class functions of G, $\dim_{\mathbb{C}} \mathcal{C}(G) = k$, with basis $\chi_1, \ldots \chi_k$, the irreducible characters of G.

- $(f_1 + f_2)(g) = f_1(g) + f_2(g)$
- $(f_1 f_2)(g) = f_1(g) f_2(g)$
- $\exists$ involution (homomorphism of order 2) $f \mapsto f^*$ where $f^*(g) = f(g^{-1})$
- $\exists$ inner product $\langle\, , \rangle$

Duality

(9.1) Lemma. Let $\rho : G \to GL(V)$ be a representation over F and let $V^* = \mathrm{Hom}_F(V, F)$, the **dual space** of V.

Then V^* is a G-space under $\rho^*(g)\phi(v) = \phi\big(\rho(g^{-1})v\big)$, the **dual representation** of ρ. Its character is $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$.

Proof.
$$\begin{aligned} \rho^*(g_1)\big(\rho^*(g_2)\phi\big)(v) &= \big(\rho^*(g_2)\phi\big)\big(\rho(g_1^{-1})v\big) = \phi\big(\rho(g_2^{-1})\rho(g_1^{-1})v\big) \\ &= \phi\big(\rho(g_1 g_2)^{-1}(v)\big) = \big(\rho^*(g_1 g_2)\phi\big)(v) \end{aligned}$$

Character. Fix $g \in G$ and let $e_1, \ldots, e_n$ be a basis of V of eigenvectors of $\rho(g)$, say $\rho(g)e_j = \lambda_j e_j$. let $\varepsilon_1, \ldots, \varepsilon_n$ be the dual basis.

Then $\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j$, for $\big(\rho^*(g)\varepsilon_j\big)(e_i) = \varepsilon_j\big(\rho(g^{-1})e_i\big) = \varepsilon_j \lambda_j^{-1} e_i = \lambda_j^{-1}\varepsilon_j e_i$ for all i.

Hence $\chi_{\rho^*}(g) = \sum \lambda_j^{-1} = \chi_\rho(g^{-1})$. □

(9.2) Definition. $\rho : G \to GL(V)$ is **self-dual** if $V \cong V^*$ (as an isomorphism of G-spaces). Over $F = \mathbb{C}$, this holds iff $\chi_\rho(g) = \chi_\rho(g^{-1})$, and since this $= \overline{\chi_\rho(g)}$, it holds iff $\chi_\rho(g) \in \mathbb{R}$ for all g.

Example. All irreducible representations of S_n are self-dual: the conjugacy classes are determined by cycle types, so g, g^{-1} are always S_n-conjugate. Not always true for A_n: it's okay for A_5, but not for A_7 – see sheet 2, question 8.

Tensor Products

V and W, F-spaces, $\dim V = m$, $\dim W = n$. Fix bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ of V, W, respectively. The **tensor product space** $V \otimes W$ (or $V \otimes_F W$) is an mn-dimensional F-space with basis $\{v_i \otimes w_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. Thus:

(a) $V \otimes W = \left\{ \sum\limits_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} \lambda_{ij} v_i \otimes w_j : \lambda_{ij} \in F \right\}$, with 'obvious' addition and scalar multiplication.

(b) if $v = \sum \alpha_i v_i \in V$, $w = \sum \beta_j w_j \in W$, define $v \otimes w = \sum_{i,j} \alpha_i \beta_j (v_i \otimes w_j)$.

Note: not all elements of $V \otimes W$ are of this form. Some are combinations, e.g. $v_1 \otimes w_1 + v_2 \otimes w_2$, which cannot be further simplified.

(9.3) Lemma. (i) For $v \in V$, $w \in W$, $\lambda \in F$, have $(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$

(ii) If $x, x_1, x_2 \in V$ and $y, y_1, y_2 \in W$, then $(x_1 + x_2) \otimes y = (x_1 \otimes y) + (x_2 \otimes y)$ and $x \otimes (y_1 + y_2) = (x \otimes y_1) + (x \otimes y_2)$.

Hence $V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$ is bilinear.

Proof. (i) $v = \sum \alpha_i v_i$, $w = \sum \beta_j w_j$, then $(\lambda v) \otimes w = \sum_{i,j} (\lambda \alpha_i) \beta_j v_i \otimes w_j$, and $\lambda(v \otimes w) = \lambda \sum_{i,j} \alpha_i \beta_j v_i \otimes w_j$, and $v \otimes (\lambda w) = \sum_{i,j} \alpha_i (\lambda \beta_j) v_i \otimes w_j$.

All three are equal. (ii) is similar. □

(9.4) Lemma. If $\{e_1, \dots, e_m\}$ is a basis of V and $\{f_1, \dots, f_n\}$ is a basis of W, then $\{e_i \otimes f_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is a basis of $V \otimes W$.

Proof. Writing $v_k = \sum_i \alpha_{ik} e_i$, $w_\ell = \sum_j \beta_{jl} f_j$, we have $v_k \otimes w_\ell = \sum_{i,j} \alpha_{ik} \beta_{jl} (e_i \otimes f_j)$, hence $\{e_i \otimes f_j\}$ spans $V \otimes W$, and since there are mn of them, they are a basis. □

(9.5) Digression. (Tensor products of endomorphisms.) If $\alpha : V \to V$, $\beta : W \to W$ are linear endomorphisms, define $\alpha \otimes \beta : V \otimes W \to V \otimes W$, $v \otimes w \mapsto \alpha(v) \otimes \beta(w)$, and extend linearly on a basis.

Example. Given bases $\mathcal{A} = \{e_1, \dots, e_m\}$ of V, and $\mathcal{B} = \{f_1, \dots, f_n\}$ of W, if $[\alpha]_{\mathcal{A}} = A$ and $[\beta]_{\mathcal{B}} = B$, then ordering the basis $\mathcal{A} \otimes \mathcal{B}$ lexicographically (i.e., $e_1 \otimes f_1$, $e_1 \otimes f_2$, …, $e_1 \otimes f_n$, $e_2 \otimes f_1$, …, $e_m \otimes f_n$), we have

$$[\alpha \otimes \beta]_{\mathcal{A} \otimes \mathcal{B}} = \begin{bmatrix} [a_{11}B] & [a_{12}B] & \dots \\ [a_{21}B] & [a_{22}B] & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

(9.6) Proposition. Let $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ be representations of G. Define $\rho \otimes \rho' : G \to GL(V \otimes V')$ by

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij} v_i \otimes w_j \mapsto \sum \lambda_{ij} \rho(g) v_i \otimes \rho'(g) w_j$$

Then $\rho \otimes \rho'$ is a representation of G, with character $\chi_{\rho \otimes \rho'}(g) = \chi_\rho(g) \chi_{\rho'}(g)$ for all $g \in G$.

Hence the product of two characters of G is also a character of G. Note: example sheet 2, question 2, says that if ρ is irreducible and ρ' is degree 1, then $\rho \otimes \rho'$ is irreducible. if ρ' is not of degree 1, then this is usually *false*, since $\rho \otimes \rho'$ is usually reducible.

Proof. It's clear that $(\rho \otimes \rho')(g) \in GL(V \otimes V')$ for all g, and so $\rho \otimes \rho'$ is a homomorphism $G \to GL(V \otimes V')$.

Let $g \in G$. Let $v_1, \dots, v_m$ be a basis of V of eigenvectors of $\rho(g)$, and $w_1, \dots, w_n$ be a basis of V' of eigenvectors of $\rho'(g)$. So $\rho(g) v_j = \lambda_j v_j$, $\rho'(g) w_j = \mu_j w_j$.

Then $(\rho \otimes \rho')(g)(v_i \otimes w_j) = \rho(g) v_i \otimes \rho'(g) w_j = \lambda_i v_i \otimes \mu_j w_j = (\lambda_i \mu_j)(v_i \otimes w_j)$.

So $\chi_{\rho \otimes \rho'}(g) = \displaystyle\sum_{i,j} \lambda_i \mu_j = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} \mu_j = \chi_\rho(g) \chi_{\rho'}(g)$. □

Take $V = V'$ and define $V^{\otimes 2} = V \otimes V$. Let $\tau : \sum \lambda_{ij} v_i \otimes v_j \mapsto \sum \lambda_{ij} v_j \otimes v_i$, a linear G-endomorphism of $V^{\otimes 2}$ such that $\tau^2 = 1$. Therefore, eigenvalues $= \pm 1$.

(9.7) Definition. The **symmetric square** of V is $S^2V = \{x \in V^{\otimes 2} : \tau(x) = x\}$.

The **exterior square** of V is $\Lambda^2 V = \{x \in V^{\otimes 2} : \tau(x) = -x\}$. (Also called the **anti-symmetric square**, or **wedge**.)

(9.8) Lemma. S^2V and Λ^2V are G-subspaces of $V^{\otimes 2}$, and $V^{\otimes 2} = S^2V \oplus \Lambda^2 V$.

S^2V has a basis $\{v_iv_j := v_i \otimes v_j + v_j \otimes v_i, 1 \leqslant i \leqslant j \leqslant n\}$, so $\dim S^2V = \frac{1}{2}n(n+1)$.

Λ^2V has a basis $\{v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i, 1 \leqslant i < j \leqslant n\}$, so $\dim \Lambda^2 V = \frac{1}{2}n(n-1)$.

Proof. Elementary linear algebra.

To show $V^{\otimes 2}$ is reducible, write $x \in V^{\otimes 2}$ as $x = \underbrace{\tfrac{1}{2}(x + \tau(x))}_{\in S^2} + \underbrace{\tfrac{1}{2}(x - \tau(x))}_{\in \Lambda^2}$. □

(9.9) Lemma. If $\rho : G \to GL(V)$ is a representation affording character χ, then $\chi^2 = \chi_S + \chi_\Lambda$ where χ_S $(= S^2\chi)$ is the character of G on the subrepresentation on S^2V, and χ_Λ $(= \Lambda^2\chi)$ is the character of G on the subrepresentation on $\Lambda^2 V$.

Moreover, for $g \in G$, $\chi_S(g) = \frac{1}{2}\big(\chi^2(g) + \chi(g^2)\big)$ and $\chi_\Lambda(g) = \frac{1}{2}\big(\chi^2(g) - \chi(g^2)\big)$.

Proof. Compute characters χ_S, χ_Λ. Fix $g \in G$. Let $v_1, \ldots, v_m$ be a basis of V of eigenvectors of $\rho(g)$, say $\rho(g)v_i = \lambda_i v_i$. Then $g\, v_iv_j = \lambda_i\lambda_j v_iv_j$ and $g\, v_i \wedge v_j = \lambda_i\lambda_j v_i \wedge v_j$.

Hence $\chi_S(g) = \displaystyle\sum_{1 \leqslant i \leqslant j \leqslant n} \lambda_i\lambda_j$ and $\chi_\Lambda(g) = \displaystyle\sum_{1 \leqslant i < j \leqslant n} \lambda_i\lambda_j$.

Now $\big(\chi(g)\big)^2 = \Big(\sum \lambda_i\Big)^2 = \sum \lambda_i^2 + 2\displaystyle\sum_{i<j} \lambda_i\lambda_j = \chi(g^2) + 2\chi_\Lambda(g)$.

So, $\chi_\Lambda(g) = \frac{1}{2}\big(\chi^2(g) - \chi(g^2)\big)$, and so $\chi_S(g) = \frac{1}{2}\big(\chi^2(g) + \chi(g^2)\big)$, as $\chi^2 = \chi_S + \chi_\Lambda$. □

'Usual trick to find characters: diagonalise and hope for the best!'

Example. $G = S_5$ (again)

	1	15	20	24	10	20	30	$\leftarrow \lvert\mathcal{C}_j\rvert$
	1	(12)(34)	(123)	(12345)	(12)	(123)(45)	(1234)	$\leftarrow g_j$
$1_G = \chi_1$	1	1	1	1	1	1	1	
sign $= \chi_2$	1	1	1	1	-1	-1	-1	
$\pi_{X_5} - 1 = \chi_3$	4	0	1	-1	2	-1	0	
$\chi_3\chi_2 = \chi_4$	4	0	1	-1	-2	1	0	
$S^2\chi - 1 - \chi_3 = \chi_5$	5	1	-1	0	-1	-1	1	
$\chi_5\chi_2 = \chi_6$	5	1	-1	0	1	1	-1	
$\Lambda^2\chi = \chi_7$	6	-2	0	1	0	0	0	

To find χ_5 and χ_7, we use (9.9) on $\chi_4 = \chi$.

	1	(12)(34)	(123)	(12345)	(12)	(123)(45)	(1234)
$\chi^2(g)$	16	0	1	1	4	1	0
$\chi(g^2)$	4	4	1	-1	4	1	0
$\chi_S(g)$	10	2	1	0	4	1	0
$\chi_\Lambda(g)$	6	-2	0	1	0	0	0

We have seen χ_S already as $\pi_{\binom{5}{2}}$. Check inner product $= 3$; contains $1, \chi_3$.

Characters of $G \times H$ (cf. (4.5) for abelian groups)

(9.10) Proposition. If G, H are finite groups, with irreducible characters $\chi_1, \ldots, \chi_k$ and $\psi_1, \ldots, \psi_\ell$ respectively, then the irreducible characters of the direct product $G \times H$ are precisely $\{\chi_i \psi_j : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l\}$ where $\chi_i \psi_j(g, h) = \chi_i(g)\psi_j(h)$.

Proof. If $\rho : G \to GL(V)$ affording χ and $\rho' : H \to GL(W)$ affording ψ, then $\rho \otimes \rho' : G \times H \to GL(V \otimes W)$, $(g, h) \mapsto \rho(g) \otimes \rho'(h)$ is a representation of $G \times H$ on $V \otimes W$ by (9.6). And $\chi_{\rho \otimes \rho'} = \chi \psi$, also by (9.6).

Claim: $\chi_i \psi_j$ are distinct and irreducible, for:

$$\begin{aligned}
\langle \chi_i \psi_j, \chi_r \psi_s \rangle_{G \times H} &= \frac{1}{|G \times H|} \sum_{(g,h)} \overline{\chi_i \psi_j(g, h)}\, \chi_r \psi_s(g, h) \\
&= \left(\frac{1}{|G|} \sum_g \overline{\chi_i(g)}\, \chi_r(g) \right) \left(\frac{1}{|H|} \sum_h \overline{\psi_j(h)}\, \psi_s(h) \right) \\
&= \delta_{ir} \delta_{js}
\end{aligned}$$

Complete set: $\sum_{i,j} \chi_i \psi_j(1)^2 = \sum_i \chi_i^2(1) \sum_j \psi_j^2(1) = |G|\,|H| = |G \times H|$. $\square$

Exercise. $D_6 \times D_6$ has 9 characters.

Digression: a general approach to tensor products

V, W, F-spaces (general F, even a non-commutative ring).

(9.11) Definition. $V \otimes W$ is the F-space with a bilinear map $t : V \times W \to T$, $(v, w) \mapsto v \otimes w =: t(v, w)$, such that any bilinear $f : V \times W \to X$ (X any F-space) can be 'factored through' it:

$$\begin{tikzcd}
V \times W \arrow[r, "t"] \arrow[dr, "f"'] & T \arrow[d, "\exists f'"] \\
 & X
\end{tikzcd}$$

I.e., there exists linear $f' : T \to X$ such that $f' \circ t = f$.

This is the **universal property** of the tensor product.

Claim. Such T exists and is unique up to isomorphism.

Existence. Take (huge) space M with basis $\{(v, w) : v \in V, w \in W\}$. Factor out the subspace N generated by 'all the things you want to be zero', i.e. by

$$\left.\begin{array}{l}
(v_1 + v_2, w) - (v_1, w) - (v_2, w) \\
(v, w_1 + w_2) - (v, w_1) - (v, w_2) \\
(\lambda v, w) - \lambda(v, w), \quad (v, \lambda w) - \lambda(v, w)
\end{array}\right\} \text{for all } v, v_1, v_2 \in V,\ w, w_1, w_2 \in W,\ \lambda \in F.$$

Define t to be the map embedding $V \times W \to M$ followed by the natural quotient map

$$\begin{tikzcd}
V \times W \arrow[r, "t"] \arrow[dr, "f"'] & M/N \arrow[d, "\exists f'"] \\
 & X
\end{tikzcd}$$

Check t is bilinear (we've quotiented out the relevant properties to make it so). f' is defined on our basis of M, $(v, w) \mapsto f(v, w)$, extended linearly. $f' = 0$ on all elements of N, hence well-defined on M/N.

Uniqueness.

$$\begin{tikzcd} V \times W \arrow[r] \arrow[dr] & T \arrow[d, shift left] \\ & T' \arrow[u, shift left] \end{tikzcd}$$

Apply universal property with respect to T, T'. Linear maps give isomorphism.

□

Henceforth, we think of $V \otimes W$ as being generated by elements $v \otimes w$ ($v \in V, w \in W$) and satisfying

$$\begin{aligned} (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ \lambda(v \otimes w) &= \lambda v \otimes w = v \otimes \lambda w \end{aligned}$$

(9.12) Lemma. If $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ are bases of V, W respectively then $\{e_i \otimes f_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is a basis of $V \otimes W$.

Proof. (Span.) Any $v \otimes w$ can be expressed (hence so can any element of $V \otimes W$) as $v = \sum_i \alpha_i e_i$, $w = \sum_j \beta_j f_j \Rightarrow v \otimes w = \sum_{i,j} \alpha_i \beta_j e_i \otimes f_j$.

(Independence.) Find a linear functional ϕ sending $e_i \otimes f_j$ to 1 and all the rest to 0. For, take dual basis $\{\varepsilon_i\}$, $\{\phi_j\}$ to the above. Define $\phi(v \otimes w) = \varepsilon_i(v)\phi_j(w)$ and check $\phi(e_i \otimes f_j) = 1$, other $= 0$. □

(9.13) Lemma. There is a 'natural' (basis independent) isomorphism in each of the following.

(i) $V \otimes W \cong W \otimes V$

(ii) $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$

(iii) $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$

Proof. (i) $v \otimes w \mapsto w \otimes v$ and extend linearly. It's well-defined: $(v, w) \mapsto w \otimes v$ is a bilinear map $V \times W \to W \otimes V$. So by the universal property $v \otimes w \mapsto w \otimes v$ gives a well-defined linear map.

(ii) $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$ and extend linearly. It's well-defined: fix $u \in U$, then $(v, w) \mapsto (u \otimes v) \otimes w$ is bilinear, so get $v \otimes w \mapsto (u \otimes v) \otimes w$.
Varying u, $(u, v \otimes w) \mapsto (u \otimes v) \otimes w$ is a well-defined bilinear map $U \times (V \otimes W) \to (U \otimes V) \otimes W$. Hence, get linear map $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$.

(iii) Similar. (See Telemann, chapter 6.) □

(9.14) Lemma. Let $\dim V$, $\dim W < \infty$. Then $\mathrm{Hom}(V, W) \cong V^* \otimes W$ naturally as G-spaces, if V, W are both G-spaces.

Proof. The natural map $V^* \times W \to \mathrm{Hom}(V, W)$, $(\alpha, w) \mapsto (\phi : v \mapsto \alpha(v)w)$ is bilinear, so $\alpha \otimes w \mapsto \phi$, extended linearly, is a linear map, $V^* \otimes W \to \mathrm{Hom}(V, W)$.

It's bijective as it takes basis to basis: $\varepsilon_i \otimes f_j \mapsto (E_{ji} : e_i \mapsto f_j)$. □

Returning to the proof of orthogonality at the end of chapter 6: the missing link was to observe that $U = \mathrm{Hom}(V', V) \cong (V')^* \otimes V$, hence $\chi_n(g) = \chi_{(V')^* \otimes V}(g) = \chi_{V'}(g^{-1})\chi_V(g)$.

Symmetric and exterior powers

V an F-space, $\dim V = d$, basis $\{e_1, \ldots, e_d\}$, $n \in \mathbb{N}$. Then $V^{\otimes n} = V \otimes \ldots \otimes V$ (n times), of dimension d^n.

Note, S_n acts on $V^{\otimes n}$: for $\sigma \in S_n$, $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$, and extend linearly ('place permutations').

The S_n-action commutes with any G-action on V.

(9.15) Definition.
The **symmetric powers**, $S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \text{ for all } \sigma \in S_n\}$.
The **exterior powers**, $\Lambda^n V = \{x \in V^{\otimes n} : \sigma(x) = \operatorname{sgn}(\sigma) x \text{ for all } \sigma \in S_n\}$.

These are G-subspaces of $V^{\otimes n}$, but if $n > 2$ then there are others obtained from the S_n-action.

Exercises. Basis for $S^n V$ is $\left\{ \frac{1}{n!} \sum_{\sigma \in S_n} v_{i_{\sigma(1)}} \otimes \ldots \otimes v_{i_{\sigma(n)}} : 1 \leqslant i_1 \leqslant \ldots \leqslant i_n \leqslant d \right\}$.

Basis for $\Lambda^n V$ is $\left\{ \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) v_{i_{\sigma(1)}} \otimes \ldots \otimes v_{i_{\sigma(n)}} : 1 \leqslant i_1 < \ldots < i_n \leqslant d \right\}$.

So $\dim S^n V = \binom{d+n-1}{n}$ and $\dim \Lambda^n V = \binom{d}{n}$.

(9.16) Definition. Let $T^n V = V^{\otimes n} = V \otimes \ldots \otimes V$.

The **tensor algebra** of V is $T(V) = \bigoplus_{n \geqslant 0} T^n V$, where $T^0 V = \{0\}$ – an F-space with obvious addition and scalar multiplication.

There is a product: for $x \in T^n V$, $y \in T^m V$, get $x.y := x \otimes y \in T^{n+m} V$, thus giving a **graded algebra** (with product $T^n V \otimes T^m V \to T^{n+m} V$).

Finally, define:

$S(V) = T(V)/(\text{ideal generated by } u \otimes v - v \otimes u)$ – the **symmetric algebra**,
$\Lambda(V) = T(V)/(\text{ideal generated by } v \otimes v)$ – the **exterior algebra**.

Character ring

$\mathcal{C}(G)$ is a ring, so the sum and product of characters are class functions. This chapter verified that they are in fact characters afforded by the sum and tensor product of their corresponding representations.

(9.17) The $\mathbb{Z}$-submodule of $\mathcal{C}(G)$ spanned by the irreducible characters of G is the **character ring** of G, written $R(G)$.

Elements of $R(G)$ are called difference/generalised/virtual characters.

$\psi \in R(G) : \psi = \sum_{\chi \text{ irred}} n_\chi \chi$, $n_\chi \in \mathbb{Z}$.

$R(G)$ is a commutative ring, and any generalised character is a difference of two characters.

I.e., $\psi = \alpha - \beta$, α, β characters, where $\alpha = \sum_{n_\chi \geqslant 0} n_\chi \chi$, $\beta = -\sum_{n_\chi < 0} n_\chi \chi$.

The $\{\chi_i\}$ form a $\mathbb{Z}$-basis for $R(G)$, as free $\mathbb{Z}$-module.

Suppose $\alpha \in R(G)$ and $\langle \alpha, \alpha \rangle = 1$. Then $\alpha(1) > 0$ implies α is the character of an irreducible representation of G.

Proof. Let the irreducible characters of G be $\chi_1, \ldots, \chi_k$. Then $\alpha = \sum n_i \chi_i$. So $\langle \alpha, \alpha \rangle = \sum n_i^2 = 1$, so exactly one $n_i = \pm 1$. But $\alpha(1) > 0$, so one $n_i = 1$ and the rest are 0.

Henceforth we don't distinguish between a character and its negative, and we often study generalised characters of norm 1 ($\langle \alpha, \alpha \rangle = 1$) rather than just irreducible characters.

10. Induction and Restriction

Throughout, $H \leqslant G$.

(10.1) Definition (Restriction). Let $\rho : G \to GL(V)$ be a representation affording χ. Can think of V as a H-space by restricting attention to $h \in H$.

Get $\operatorname{Res}^G_H \rho : H \to GL(V)$, the **restriction of ρ to H**. (Also written $\rho|_H$ or ρ_H.)

It affords the character $\operatorname{Res}^G_H \chi$. (Also written $\chi|_H$ or χ_H.)

(10.2) Lemma. If ψ is any non-zero character of H, then there exists an irreducible character χ of G such that

$$\left.\begin{array}{l} \bullet\ \psi \subset \operatorname{Res}^G_H \chi \\ \bullet\ \psi \text{ is a } \textbf{constituent} \text{ of } \operatorname{Res}^G_H \chi \\ \bullet\ \langle \operatorname{Res}^G_H \chi, \psi \rangle \neq 0 \end{array}\right\} \text{ 3 ways of saying the same thing}$$

Proof. List the irreducible characters of G: $\chi_1, \ldots \chi_k$. Recall χ_{reg} from (5.9).

$$0 \neq \frac{|G|}{|H|}\,\psi(1) = \langle \chi_{\text{reg}}|_H, \psi \rangle_H = \sum \deg \chi_i \langle \chi_i|_H, \psi \rangle_H$$

Therefore $\langle \chi_i|_H, \psi \rangle \neq 0$ for some i. $\square$

(10.3) Lemma. Let χ be an irreducible character of G, and let $\operatorname{Res}^G_H \chi = \sum_i c_i \chi_i$ with χ_i irreducible characters of H, where $c_i \in \mathbb{Z}_{\geqslant 0}$.

Then $\sum c_i^2 \leqslant |G : H|$, with equality iff $\chi(g) = 0$ for all $g \in G \setminus H$.

Proof. $\sum c_i^2 = \langle \operatorname{Res}^G_H \chi, \operatorname{Res}^G_H \chi \rangle_H = \dfrac{1}{|H|} \displaystyle\sum_{h \in H} |\chi(h)|^2.$

$$\begin{aligned} \text{But } 1 = \langle \chi, \chi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\ &= \frac{1}{|G|} \left(\sum_{h \in H} |\chi(h)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2 \right) \\ &= \frac{|H|}{|G|} \sum c_i^2 + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2}_{\geqslant 0, \text{ and } = 0 \Leftrightarrow \chi(g) = 0\ \forall g \in G \setminus H} \end{aligned}$$

Therefore $\sum c_i^2 \leqslant |G : H|$, with equality iff $\chi(g) = 0$ for all $g \in G \setminus H$. $\square$

Example. $G = S_5$, $H = A_5$, $\psi_i = \operatorname{Res}^G_H \chi_i$.

$$\begin{array}{lcccc} & & \neq 0 \text{ somewhere outside } H & & \chi(g)=0\ \forall g \in S_5 \setminus A_5 \\ & & \searrow & & \swarrow \\ \deg \chi_i & 1\quad 1 & 4\quad 4 & 5\quad 5 & 6 \\ & \backslash\ / & \backslash\ / & \backslash\ / & /\ \backslash \\ \deg \psi_i & 1 & 4 & 5 & \underbrace{3\quad 3} \end{array}$$

general fact about normal subgroups:
splits into constituents of equal degree
(Clifford's Theorem)

(10.4) Definition (Induction). If ψ is a class function of H, define

$$\psi^G = \operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(x^{-1}gx), \quad \text{where} \quad \mathring{\psi}(y) = \begin{cases} \psi(y) & y \in H \\ 0 & y \notin H \end{cases}.$$

(10.5) Lemma. If ψ is a class function of H, then $\operatorname{Ind}_H^G \psi$ is a class function of G, and $\operatorname{Ind}_H^G \psi(1) = |G:H|\,\psi(1)$.

Proof. Clear, noting that $\operatorname{Ind}_H^G \psi(1) = \dfrac{1}{|H|} \displaystyle\sum_{x \in G} \mathring{\psi}(1) = |G:H|\,\psi(1)$. $\square$

Let $n = |G:H|$. Let $t_1 (= 1), t_2, \ldots, t_n$ be a **left transversal** of H in G (i.e., a complete set of coset representatives), so $t_1 H (= H)$, $t_2 H$, $\ldots$, $t_n H$ are precisely the left cosets of H in G.

(10.6) Lemma. Given a transversal as above, $\operatorname{Ind}_H^G \psi(g) = \displaystyle\sum_{i=1}^{n} \mathring{\psi}(t_i^{-1} g t_i)$.

Proof. For $h \in H$, $\mathring{\psi}\big((t_i h)^{-1} g (t_i h)\big) = \mathring{\psi}(t_i^{-1} g t_i)$, as ψ is a class function of H. $\square$

(10.7) Theorem (Frobenius Reciprocity). ψ a class function on H, ϕ a class function on G. Then

$$\langle \operatorname{Res}_H^G \phi, \psi \rangle_H = \langle \phi, \operatorname{Ind}_H^G \psi \rangle_G.$$

Proof.

$$\begin{aligned}
\langle \phi, \psi^G \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi^G(g) \\
&= \frac{1}{|G|\,|H|} \sum_{g, x \in G} \overline{\phi(g)}\, \mathring{\psi}(x^{-1}gx) \\
&= \frac{1}{|G|\,|H|} \sum_{x, y} \overline{\phi(y)}\, \mathring{\psi}(y) \qquad \text{(put } y = x^{-1}gx\text{)} \\
&= \frac{1}{|H|} \sum_{y \in G} \overline{\phi(y)}\, \mathring{\psi}(y) \qquad \text{(independent of } x\text{)} \\
&= \frac{1}{|H|} \sum_{y \in H} \overline{\phi(y)} \psi(y) \\
&= \langle \phi_H, \psi \rangle_H
\end{aligned}$$

$\square$

(10.8) Corollary. If ψ is a character of H then $\operatorname{Ind}_H^G \psi$ is a character of G.

Proof. Let χ be an irreducible character of G.

By (10.7), $\langle \operatorname{Ind}_H^G \psi, \chi \rangle_G = \langle \psi, \operatorname{Res}_H^G \chi \rangle \in \mathbb{Z}_{\geqslant 0}$, since ψ, $\operatorname{Res}_H^G \chi$ are characters.

Hence $\operatorname{Ind}_H^G \psi$ is a linear combination of irreducible characters, with positive coefficients, hence a character. $\square$

(10.9) Lemma. Let ψ be a character (or even a class function) of $H \leqslant G$ and let $g \in G$. Let $\mathcal{C}_G(g) \cap H = \bigcup_{i=1}^{m} \mathcal{C}_H(x_i)$ (disjoint union), where x_i are representatives of the m H-conjugacy classes of elements of H conjugate to g.

Then, for $m = 0$, $\operatorname{Ind}_H^G \psi(g) = 0$, and for $m \geqslant 1$, $\operatorname{Ind}_H^G \psi(g) = |C_G(g)| \displaystyle\sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|}$.

Diversion. Let $H, K \leqslant G$. A **double coset** of H and K in G is a set of the form $HxK = \{hxk : h \in H, k \in K\}$ for some $x \in G$.

Facts. Two double cosets are either disjoint or equal, and

$$|HxK| = \frac{|H|\,|K|}{|H \cap xKx^{-1}|} = \frac{|H|\,|K|}{|xHx^{-1} \cap K|}.$$

Proof of (10.9). If $m = 0$ then $\{x \in G : x^{-1}gx \in H\} = \emptyset$, and then $\operatorname{Ind}_H^G \psi(g) = 0$ by definition.

Let $m > 0$. Let $X_i = \{x \in G : x^{-1}gx \in H$ and is conjugate in H to $x_i\}$, for $1 \leqslant i \leqslant m$. The X_i are pairwise disjoint and their union is $\{x \in G : x^{-1}gx \in H\}$.

By definition,

$$\begin{aligned}
\operatorname{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{i=1}^{m} \sum_{x \in X_i} \psi(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{i=1}^{m} \sum_{x \in X_i} \psi(x_i) \\
&= \sum_{i=1}^{m} \frac{|X_i|}{|H|} \psi(x_i).
\end{aligned}$$

We need to calculate $\dfrac{|X_i|}{|H|}$.

Fix $1 \leqslant i \leqslant m$ and choose $g_i \in G$ such that $g_i^{-1}gg_i = x_i$. So for all $c \in C_G(g)$ and $h \in H$ we have

$$(cg_ih)^{-1}g(cg_ih) = h^{-1}g_i^{-1}c^{-1}gcg_ih = h^{-1}g_i^{-1}gg_ih = h^{-1}x_ih,$$

i.e. $cg_ih \in X_i$, and hence $C_G(g)g_iH \subset X_i$.

Conversely, for $x \in X_i$, we have $x^{-1}gx = h^{-1}x_ih = h^{-1}(g_i^{-1}gg_i)h$ for some $h \in H$. So $xh^{-1}g_i^{-1} \in C_G(g)$, and hence $x \in C_G(g)g_ih \subset C_G(g)g_iH$.

Thus $C_G(g)g_iH = X_i$, so $|X_i| = |C_G(g)g_iH| = \dfrac{|C_G(g)|\,|H|}{|H \cap g_i^{-1}C_G(g)g_i|}$.

But $g_i^{-1}C_G(g)g_i = C_G(g_i^{-1}gg_i) = C_G(x_i)$.

So $|X_i| = |H : H \cap C_G(x_i)|\,|C_G(g)| = |H : C_H(x_i)|\,|C_G(g)|$.

So $\dfrac{|X_i|}{|H|} = \dfrac{|C_G(g)|}{|C_H(x_i)|}$, giving the result. $\square$

(10.10) Lemma. If $\psi = 1_H$, the principal character of H, then $\operatorname{Ind}_H^G 1_H = \pi_X$, the permutation character of G on the set X of left cosets of H in G.

Proof.

$$\begin{aligned}
\operatorname{Ind}_H^G 1_H(g) &= \sum \mathring{1}_H(t_i^{-1}gt_i) && \text{(where the } t_i \text{ form a transversal)}\\
&= \left|\{i : t_i^{-1}gt_i \in H\}\right| \\
&= \left|\{i : g \in t_iHt_i^{-1}\}\right| && \leftarrow \text{stabiliser in } G \text{ of the point } t_iH \in X\\
&= |\mathrm{fix}_X(g)| \;=\; \pi_X && \text{(see (7.1))}
\end{aligned}$$

□

Remark. Recalling (7.3):

$$\langle \pi_X, 1_G\rangle_G \underset{(10.10)}{=} \langle \operatorname{Ind}_H^G 1_H, 1_G\rangle_G \underset{(10.7)}{=} \langle 1_H, \operatorname{Res}_H^G 1_G\rangle_H = \langle 1_H, 1_H\rangle_H = 1.$$

Examples. (a) Recall (7.9), $G = S_5$ acting on $X =$ the set of Sylow 5-subgroups of G. $\pi_X = \operatorname{Ind}_H^G 1_H$, where $H = \langle (12345), (2354)\rangle$. Note $|H| = 20$.

H-ccls	1	(12345)	(2354)	(2453)	(25)(34)
size	1	4	5	5	5

G-ccls	1	(12)(34)	(123)	(12345)	(12)	(123)(45)	(1234)
size	1	15	20	24	10	20	30

$\pi_X(2354) = \frac{120}{30}(\frac{5}{20} + \frac{5}{20}) = 4(\frac{1}{4} + \frac{1}{4}) = 2$

$\pi_X((25)(34)) = \frac{120}{15}(\frac{5}{20}) = 8(\frac{1}{4}) = 2$. (All elements (25)(34) are conjugate in H.)

(b) Recall (2.17) and (7.8). $H = C_4 = \langle (1234)\rangle \leqslant G = S_4$, index 6.
Character of induced representation $\operatorname{Ind}_{C_4}^{S_4}(\alpha)$, where α is faithful 1-dimensional representation of C_4. If $\alpha((1234)) = i$ then character of α is:

	1	(1234)	(13)(24)	(1432)
χ_α	1	i	-1	$-i$

Induced representations:

size	1	6	8	3	6
ccls	1	(12)	(123)	(12)(34)	(1234)
$\operatorname{Ind}_{C_4}^{S_4}(\alpha)$	6	0	0	-2	0

For (12)(34), only one of 3 elements in S_4 that it's conjugate to lies in H. So $\operatorname{Ind}_H^G(\alpha) = 8(-\frac{1}{4}) = -2$.

(1234) is conjugate to 6 elements of S_4, of which 2 are in C_4 (viz. (1234), (1432)). So $\operatorname{Ind}_H^G(\alpha) = 4(\frac{i}{4} - \frac{i}{4}) = 0$.

Induced modules

$H \leqslant G$, index n. $t_1 = 1, t_2, \ldots, t_n$ a transversal – i.e. $H, t_2H, \ldots, t_nH$ are the left cosets of H in G. Let W be an H-space.

(10.11) Definition. Let $V = W \oplus t_2 \otimes W \oplus \ldots \oplus t_n \otimes W$, where $t_i \otimes W = \{t_i \otimes w : w \in W\}$. ('Essentially tensored group algebra with W.')

So $\dim V = n \dim W$ and we write $V = \operatorname{Ind}_H^G W$.

***G*-action.** $g \in G$, for all i, there exists a unique j with $t_j^{-1}gt_i \in H$ (namely t_jH is the unique coset which contains gt_i).

Define $g(t_i w) = t_j\big((t_j^{-1} g t_i)w\big)$. (Drop the $\otimes$s, so $t_i w := t_i \otimes w$.) Check this is a G-action:

$$\begin{aligned}
\underbrace{g_1\,(g_2 t_i w)}_{(\exists \text{ unique } j \text{ s.t. } g_2 t_1 H = t_j H)} &= g_1\big(t_j(t_j^{-1} g_2 t_i)w\big) \\
&= \underbrace{t_\ell\big((t_\ell^{-1} g_1 t_j)}_{(\exists \text{ unique } \ell \text{ s.t. } g_1 t_j H \in t_\ell H)}(t_j^{-1} g_2 t_i)w\big) \\
&= t_\ell\big(t_\ell^{-1}(g_1 g_2)t_i\big)w \\
&= (g_1 g_2)(t_i w)
\end{aligned}$$

ℓ is unique with $(g_1 g_2)t_i H \in t_\ell H$.

It has the right character (still dropping the $\otimes$) $g : t_i w \mapsto t_j\underbrace{(t_j^{-1} g t_i)}_{\in W} w$.

So the contribution to the character is 0 unless $j = i$, i.e. unless $t_i^{-1} g t_i \in H$, then it contributes $\psi(t_i^{-1} g t_i)$, i.e. $\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^n \mathring{\psi}(t_i^{-1} g t_i)$, thus agreeing with (10.6).

Example. Module-theoretic version of (10.10) states: $\operatorname{Ind}_H^G(\mathbb{C}) = \mathbb{C}X$, where $X = G/H$. In particular, $\operatorname{Ind}_1^G(\mathbb{C}) = \rho_{\text{reg}}$.

Remarks (non-examinable). (1) There is also a 'Frobenius reciprocity' for modules: for W a H-space, V a G-space, $\operatorname{Hom}_H(W, \operatorname{Res}_H^G V) \cong \operatorname{Hom}_G(\operatorname{Ind}_H^G W, V)$ naturally, as vector spaces.

This is an example of a 'Nakayama relation'. See Telemann 15.9 – works over general fields.

(2) Tensor products of modules over rings. In (10.11), $V = FG \otimes_{FH} W$.

Replace FG by R, FH by S, and try to generalise. In general, given rings R, S, and modules U an (R,S)-bimodule and W a left S-module, then $U \otimes W$ is a left R-module with **balanced** map $t : U \times W \to U \otimes W$ such that any balanced map $f : U \times W \to X$, any left R-module X can be factored through t.

$$\begin{tikzcd}
U \times W \arrow[r, "t"] \arrow[dr, "f"'] & U \otimes W \arrow[d, "\exists \text{ unique module homomorphism} f'"] \\
& X
\end{tikzcd}$$

'Balanced' means
$$\begin{aligned}
&f(u_1 + u_2, w) = f(u_1, w) + f(u_2, w) \\
&f(u, w_1 + w_2) = f(u, w_1) + f(u, w_2) \\
&f(\lambda u, w) = f(u, \lambda w) \quad (\text{for all } \lambda \in S)
\end{aligned}$$

Then $\operatorname{Ind}_H^G W = FG \otimes W$ is now a well-defined FG-module, since W is a left FH-module, FG is (FG, FH)-bimodule. (Alperin-Bell.)

11. Frobenius Groups

(11.1) Frobenius Theorem (1891). G a transitive permutation group on a set X, with $|X| = n$. Assume that each non-identity element of G fixes at most one element of X. Then

$$K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \text{ for all } \alpha \in X\}$$

is a normal subgroup of G of order n.

Proof. (Suzuki, Collins (book).) Required to prove $K \trianglelefteq G$.

Let $H = G_\alpha$ (stabiliser of $\alpha \in X$), so conjugates of H are the stabilisers of single elements of X, as $G_{g\alpha} = gG_\alpha g^{-1}$. No two conjugates can share a non-identity element (hypothesis).

So H has n distinct conjugates and G has $n(|H| - 1)$ elements that fix exactly one element of X. But $|G| = |X|\,|H| = n\,|H|$. (X and G/H are isomorphic G-sets, as the action is transitive), hence $|K| = |G| - n(|H| - 1) = n$.

Let $1 \neq h \in H$. Suppose $h = gh'g^{-1}$, some $g \in G, h' \in H$. Then h lies in $H = G_\alpha$ and $gHg^{-1} = G_{g\alpha}$. By hypothesis, $g\alpha = \alpha$, hence $g \in H$. So $H \cap \mathrm{ccl}_G(h)$ is precisely $\mathrm{ccl}_H(h)$.

Similarly, if $g \in C_G(h)$ then $h = ghg^{-1} \in G_{g\alpha}$ hence $g \in H$, i.e. $C_G(h) = C_H(h)$.

Every element of G lies either in K or in one of the n stabilisers, each of which is conjugate to H. So every element of $G \setminus K$ is conjugate with a non-1 element of H. So

$$\{\underbrace{1, h_2, \ldots, h_t}_{\text{reps of } H\text{-ccls}}, \underbrace{y_1, \ldots, y_u}_{\text{reps of ccls of } G \text{ comprising } K \setminus \{1\}}\}$$

is a set of conjugacy class representatives for G.

Problem. To show $K \leqslant G$.

Take $\theta = 1_G$, $\{1_H = \psi_1, \psi_2, \ldots, \psi_t\}$ irreducible characters of H. Fix some $1 \leqslant i \leqslant t$. Then if $g \in G$,

$$\operatorname{Ind}_H^G \psi_i(g) = \begin{cases} |G:H|\,\psi_i(1) = n\psi_i(1) & g = 1 \\ \psi_i(h_j) & g = h_j \ (2 \leqslant j \leqslant t) \\ 0 & g = y_k \ (1 \leqslant k \leqslant u) \end{cases}$$

$C_G(h_j) = C_H(h_j)$ & (10.9)

Fix some $2 \leqslant i \leqslant t$ and put $\theta_i = \psi_i^G - \psi_i(1)\psi_1^G + \psi_i(1)\theta_1 \in R(G)$, by (9.16).

Values for $2 \leqslant j \leqslant t$, $1 \leqslant k \leqslant u$:

	1	h_j	y_k
ψ_i^G	$n\psi_i(1)$	$\psi_i(h_j)$	0
$\psi_i(1)\psi_1^G$	$n\psi_i(1)$	$\psi_i(1)$	0
$\psi_i(1)\theta_1$	$\psi_i(1)$	$\psi_i(1)$	$\psi_i(1)$
θ_i	$\psi_i(1)$	$\psi_i(h_j)$	$\psi_i(1)$

$$
\begin{aligned}
\langle\theta_i,\theta_i\rangle &= \frac{1}{|G|}\sum_{g\in G}|\theta_i(g)|^2\\
&= \frac{1}{|G|}\Big(\sum_{g\in K}|\theta_i(g)|^2+\sum_{\alpha\in X}\sum_{1\neq g\in G_\alpha}|\theta_i(g)|^2\Big)\\
&= \frac{1}{|G|}\Big(n\psi_i^2(1)+n\sum_{1\neq h\in H}|\theta_i(h)|^2\Big)\\
&= \frac{1}{|H|}\sum|\psi_i(h)|^2\\
&= \langle\psi_i,\psi_i\rangle\\
&= 1 \quad \text{(row orthogonality of irreducible H-characters)}
\end{aligned}
$$

By (9.17) either θ_i or $-\theta_i$ is an irreducible character of G, since $\theta_i(1)>0$, it is θ_i. Let $\theta=\sum_{i=1}^t\theta_i(1)\theta_i$. Column orthogonality $\Rightarrow$ $\theta(h)=\sum_{i=1}^t\psi_i(1)\psi_i(h)=0$ $(1\neq h\in H)$ and for any $y\in K$, $\theta(y)=\sum\psi_i^2(1)=|H|$.

So $\theta(g)=\begin{cases}|H| & \text{if } g\in K\\ 0 & \text{if } g\notin K\end{cases}$

Therefore $K=\{g\in G:\theta(g)=\theta(1)\}\underset{(5.3)}{\trianglelefteq} G$. □

(11.2) Definition. A **Frobenius group** is a group G having a subgroup H such that $H\cap H^g=1$ for all $g\notin H$. H is a **Frobenius complement**.

(11.3) Any finite Frobenius group satisfies the hypothesis of (11.1). The normal subgroup K is the **Frobenius kernel** of G.

If G is Frobenius and H a complement then the action of G on G/H is faithful and transitive. If $1\neq g\in G$ fixes xH and yH then $g\in xHx^{-1}\cap yHy^{-1}$, which implies that $H\cap(y^{-1}x)H(y^{-1}x)^{-1}\neq 1$, and so $xH=yH$.

Remarks. (i) Thompson (thesis, 1959) worked on the structure of Frobenius groups – e.g. showed that K is nilpotent (i.e., K is the direct product of its Sylow subgroups).

(ii) There is no proof of (11.1) known in which character theory is not used.

(iii*) Show that $G=K\rtimes H$, semi-direct product.

12. Mackey Theory

This describes restriction to a subgroup $K \leqslant G$ of an induced representation $\in W$. K, H are unrelated but usually we take $K = H$, in which case we can tell when $\operatorname{Ind}_H^G W$ is irreducible.

Special case: $W = 1$ (trivial H-space, dim 1). Then by (10.10) $\operatorname{Ind}_H^G 1 =$ permutation representation of G on $X = G/H$ (coset action on the set of left cosets of H in G).

Recall. If G is transitive on a set X and $H = G_\alpha$ ($\alpha \in X$) then the action of G on X is isomorphic to the action on G/H, viz:

(12.1) $\underbrace{g.\alpha}_{\in X} \longleftrightarrow \underbrace{gH}_{\in G/H}$ is a well-defined bijection and commutes with G-actions.

I.e., $x(g\alpha) = (xg)\alpha \longleftrightarrow x(gH) = (xg)H$.

Consider the action of G on G/H and restriction to some $K \leqslant G$. G/H splits into K-orbits; these correspond to **double cosets** $KgH = \{kgh : k \in K, h \in H\}$. The K-orbit containing gH contains precisely all kgH ($k \in K$).

(12.2) Definition. $K \setminus G/H$ is the set of double cosets KgH.

Note $|K \setminus G/H| = \langle \pi_{G/K}, \pi_{G/H} \rangle$ – see (7.4). Clearly $G_{gH} = gHg^{-1}$. Therefore $K_{gH} = gHg^{-1} \cap K$. So by (12.1) the action of K on the orbit containing gH is isomorphic to the action of K on $K/(gHg^{-1} \cap K)$.

(12.3) Proposition. $\operatorname{Res}_K^G \operatorname{Ind}_H^G 1 = \displaystyle\bigoplus_{g \in K \setminus G/H} \operatorname{Ind}_{gHg^{-1} \cap K}^K 1$,

summed over set of representatives of double cosets.

Now choose $g_1, \ldots, g_r$ such that $G = \bigcup K g_i H$. Write $H_g = gHg^{-1} \cap K \leqslant K$. Let W be an H-space, and write W_g for the H_g-space with the same underlying vector space as W of vectors, but with H_g-action from $\rho_g(x) = \rho(\underbrace{g^{-1}xg}_{\in H})$ for $x \in gHg^{-1}$.

We will prove:

(12.4) Theorem (Mackey's Restriction Formula). $\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \displaystyle\bigoplus_{g \in K \setminus G/H} \operatorname{Ind}_{H_g}^K W_g$.

In terms of characters:

(12.5) Theorem. If $\psi \in \mathcal{C}(H)$, then $\operatorname{Res}_K^G \operatorname{Ind}_H^G \psi = \displaystyle\sum_{g \in K \setminus G/H} \operatorname{Ind}_{H_g}^K \psi_g$, where ψ_g is the class function on H_g given by $\psi_g(x) = \psi(x^{g^{-1}})$.

The most useful form for applications is:

(12.6) Corollary (Mackey's Irreducibility Criterion). $H \leqslant G$, W and H-space. Then $V = \operatorname{Ind}_H^G W$ is irreducible iff

(i) W is irreducible, and

(ii) for each $g \in G \setminus H$, the two $(gHg^{-1} \cap H)$-spaces W_g and $\operatorname{Res}_{H_g}^H W$ have no irreducible constituents in common. (We say they are disjoint.)

Proof of Corollary. Take $K = H$ in (12.4), so $H_g = gHg^{-1} \cap H$. Assume W is irreducible with character ψ.

$$\begin{aligned}
\langle \operatorname{Ind}_H^G \psi, \operatorname{Ind}_H^G \psi \rangle &\underset{\text{(F.R.)}}{=} \langle \psi, \operatorname{Res}_H^G \operatorname{Ind}_H^G \psi \rangle \\
&\underset{(12.5)}{=} \sum_{g \in H \backslash G/H} \langle \psi, \operatorname{Ind}_{H_g}^H \psi_g \rangle_H \\
&\underset{\text{(F.R.)}}{=} \sum_{g \in H \backslash G/H} \langle \operatorname{Res}_{H_g}^H \psi, \psi_g \rangle_{H_g} \\
&= 1 + \sum_{\substack{g \in H \backslash G/H \\ g \notin H}} d_g \quad \text{where } d_g = \langle \operatorname{Res}_{H_g}^H \psi, \psi_g \rangle_{H_g}
\end{aligned}$$

So to get irreducibility we need all the $d_g = 0$. □

(12.7) Corollary. If $H \trianglelefteq G$, assume ψ is an irreducible character of H. Then $\operatorname{Ind}_H^G \psi$ is irreducible iff ψ is distinct from all its conjugates ψ_g for $g \in G \setminus H$, where $\psi_g(h) = \psi(h^{g^{-1}}) = \psi(g^{-1}hg)$.

Proof. Take $K = H$, so $H_g = gHg^{-1} \cap H = H$ for all g (since $H \trianglelefteq G$). ψ_g is the character of H conjugate to ψ, so $\operatorname{Res}_{H_g}^H \psi = \psi$ and the ψ_g are just the conjugates of ψ. □

Proof of (12.4). Write $V = \operatorname{Ind}_H^G W$. Fix $g \in G$, so $KgH \in K \setminus G/H$. Observe V is a direct sum of images of the form xW (officially $x \otimes W$, recall), with x running over representatives of left cosets of H in G (see (10.11)). Collect together the images xW with $x \in KgH$ (as in (12.3)) and define $V(g) = \bigoplus_{x \in KgH} xW$.

Now $V(g)$ is a K-space and $\operatorname{Res}_K^G V = \bigoplus_{\substack{g \text{ reps of} \\ K \backslash G/H}} V(g)$.

We have to prove $V(g) = \operatorname{Ind}_{H_g}^K W_g$, as K-spaces. The subgroup of K consisting of the elements x with $xgW = gW$ is $H_g = gHg^{-1} \cap K$ (see (12.2)), and $V(g) = \bigoplus_{x \in K \backslash H_g} x(gW)$.

Hence $V(g) \cong \operatorname{Ind}_{H_g}^K (gW)$.

Finally $W_g \cong gW$ as K-spaces, as the map $w \mapsto gw$ is an isomorphism. Hence the assertion. □

Examples. (a) Give a direct proof of (12.3). Hint. Write $G = \bigcup_{\substack{g_i \text{ reps of} \\ K \backslash G/H}} Kg_iH$,$(1 \leqslant i \leqslant r)$.

Let H_{g_i} have transversal $k_{i\,r_1}, \ldots, k_{i\,r_i}$ in K. Then $\{k_{ij}g_i : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant r_i\}$ is a transversal of K in G. Then compute $\operatorname{Ind}_H^G \psi(k)$.

(b) (Examples sheet 3, question 4.) $C_n \lhd D_{2n} = \langle x, y : x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle$. Mackey says that for any 1-dimensional representation α of C_n, the 2-dimensional representation $\operatorname{Ind}_{C_n}^{D_{2n}} \alpha$ is irreducible iff α is not isomorphic to α_y.
Now $y^{-1}xy = x^{-1}$, so this says that if $\alpha(x) = \zeta^i$ ($\zeta \in \mu_n$), α_g is the representation $\alpha_g(x) = \zeta^{-i}$. So for $0 < i < n/2$ (i.e. when $e^{2\pi ik/n} \neq e^{-2\pi ik/n}$) we get a 2-dimensional irreducible representation of D_{2n} this way.

13. Integrality

(13.1) Definition. $a \in \mathbb{C}$ is an **algebraic integer** if it is a root of a monic polynomial in $\mathbb{Z}[X]$. Equivalently, the subring $\mathbb{Z}[a] = \{f(a) : f(x) \in \mathbb{Z}[X]\}$ of $\mathbb{C}$ is a finitely-generated $\mathbb{Z}$-module.

Fact 1. The algebraic integers form a subring of $\mathbb{C}$. (James & Liebeck 22.3)

Fact 2. If $a \in \mathbb{C}$ is both an algebraic integer and a rational number then $a \in \mathbb{Z}$. (James & Liebeck 22.5)

Fact 3. Any subring S of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-module consists of algebraic integers.

If $s_1, \ldots, s_n$ are generators of S as a $\mathbb{Z}$-module, let $a \in S$. Then for all i, there exist $a_{ij} \in \mathbb{Z}$ with $as_i = \sum_j a_{ij}s_j$. Put $A = (a_{ij})$. Then $Av = av$, where $v = (s_1, \ldots, s_n)^t$, so a is a root of the characteristic polynomial of A. Therefore, it's an algebraic integer.

(13.2) Proposition. If χ is a character of G and $g \in G$ then $\chi(g)$ is an algebraic integer.

Corollary. There are no entries in the character table of any finite group which are rational but not integers. (Fact 2.)

Proof of (13.2). $\chi(g)$ is the sum of n^{th} roots of 1 ($n = |g|$). Each root of unity is an algebraic integer, and any sum of algebraic integers is an algebraic integer. (Fact 1.) $\square$

Recall from (2.4) the group algebra $\mathbb{C}G = \{\sum \alpha_g g : \alpha_g \in \mathbb{C}\}$ of a finite group G, the $\mathbb{C}$-space with basis the elements of G. It is also a ring.

List $\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \ldots, \mathcal{C}_k$, the G-conjugacy classes. Define the **class sums**, $C_j = \displaystyle\sum_{g \in \mathcal{C}_j} g \in \mathbb{C}G$.

$Z(\mathbb{C}G)$ is the **centre** of $\mathbb{C}G$ (not the same as $\mathbb{C}Z(G)$).

(13.3) Proposition. $C_1, \ldots, C_k$ is a basis of $Z(\mathbb{C}G)$. There exist non-negative integers a_{ijl} ($1 \leqslant i, j, l \leqslant k$) with $C_iC_j = \sum a_{ijl}C_\ell$. These are the **structure constants** for $Z(\mathbb{C}G)$.

E.g., 1, $(12) + (13) + (23)$, $(123) + (132)$ form a basis of $\mathbb{Z}(\mathbb{C}S_3)$.

Proof. $gC_jg^{-1} = C_j$, so $C_j \in Z(\mathbb{C}G)$. Clearly the C_j are linearly independent (because the conjugacy classes are pairwise disjoint).

Now suppose $z \in Z(\mathbb{C}G)$, $z = \sum_{g \in G} \alpha_g g$. Then for all $h \in G$ we have $\alpha_{h^{-1}gh} = \alpha_g$, so the function $g \mapsto \alpha_g$ is constant on G-conjugacy classes. Writing $\alpha_g = \alpha_i$ ($g \in \mathcal{C}_i$), then $z = \sum \alpha_j C_j$.

Finally $Z(\mathbb{C}G)$ is a $\mathbb{C}$-algebra ('vector space over $\mathbb{C}$ with ring multiplication'), so $C_iC_j = \sum_{l=1}^k a_{ijl}C_\ell$, as the C_j span. We claim that $a_{ijl} \in \mathbb{Z}_{\geqslant 0}$.

For: fix $g_\ell \in \mathcal{C}_\ell$, then $a_{ijl} = \#\{(x, y) \in \mathcal{C}_i \times \mathcal{C}_j : xy = g_\ell\} \in \mathbb{Z}_{\geqslant 0}$. $\square$

(13.4) Definition. Let $\rho : G \to GL(V)$ be an irreducible representation over $\mathbb{C}$ affording χ. Extend by linearity to $\rho : \mathbb{C}G \to \operatorname{End} V$, an algebra homomorphism. Such a homomorphism of algebras, $\mathbb{C}G = A \to \operatorname{End} V$ is a **representation** of A.

Let $z \in Z(\mathbb{C}G)$. Then $\rho(z)$ commutes with all $\rho(g)$ ($g \in G$), so by Schur's Lemma $\rho(g) = \lambda_z I$ for some $\lambda_z \in \mathbb{C}$. Consider the algebra homomorphism $w_\chi = w : Z(\mathbb{C}G) \to \mathbb{C}$, $z \mapsto \lambda_z$.

Then $\rho(C_i) = w(C_i)I$, so $\chi(1)w(C_i) = \sum_{g \in \mathcal{C}_i} \chi(g) = |\mathcal{C}_i|\,\chi(g_i)$ (g_i a representative of $\mathcal{C}_i$).

Therefore $w_\chi(C_i) = \dfrac{\chi(g_i)}{\chi(1)}|\mathcal{C}_i|$.

(13.5) Lemma. The values of $w_\chi(C_i) = \dfrac{\chi(g_i)}{\chi(1)}|\mathcal{C}_i|$ are algebraic integers.

Proof. Since w is an algebra homomorphism, have $w_\chi(C_i)w_\chi(C_j) = \sum_{l=1}^{k} a_{ijl} w_\chi(C_\ell)$, with $a_{ijl} \in \mathbb{Z}_{\geqslant 0}$. Thus the span $\{w(C_i) : 1 \leqslant i \leqslant k\}$ is a subring of $\mathbb{C}$, so by Fact 3 consists of algebraic integers. □

Example. Show that $a_{ijl} = \#\{(x,y) \in \mathcal{C}_i \times \mathcal{C}_j : xy = g_\ell\}$ can be obtained from the character table. In fact,

$$a_{ijl} = \frac{|G|}{|C_G(g_i)|\,|C_G(g_j)|} \sum_{s=1}^{k} \frac{\chi_s(g_i)\chi_s(g_j)\chi_s(g_\ell^{-1})}{\chi_s(1)}.$$

Hint: use column orthogonality. (See James & Liebeck 30.4.)

(13.6) Theorem. The degree of any irreducible character of G divides $|G|$.

I.e., $\chi_i(1) \big| |G|$ $(1 \leqslant i \leqslant k)$.

Proof. Given irreducible χ. ('Standard trick: show $|G|/\chi(1) \in \mathbb{N}$.')

$$\begin{aligned} \frac{|G|}{\chi(1)} &= \frac{1}{\chi(1)} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \frac{1}{\chi(1)} \sum_{i=1}^{k} |\mathcal{C}_i|\chi(g_i)\chi(g_i^{-1}) \\ &= \sum_{i=1}^{k} \frac{|\mathcal{C}_i|\chi(g_i)}{\chi(1)}\,\chi(g_i^{-1}) \end{aligned}$$

Now $\dfrac{|\mathcal{C}_i|\chi(g_i)}{\chi(1)}$ is an algebraic integer by (13.5), and $\chi(g_i^{-1})$ is a sum of roots of unity, so is an algebraic integer by (13.2)

Thus $\dfrac{|G|}{\chi(1)}$ is an algebraic integer, and since it's clearly rational, it is an integer. □

Examples. (a) If G is a p-group then $\chi(1)$ is a p-power (χ irreducible). If $|G| = p^2$ then $\chi(1) = 1$ (hence G is abelian).

(b) No simple group has an irreducible character of degree 2 (see James & Liebeck 22.13).

(c*) In fact, if χ is irreducible then $\chi(1)$ divides $|G|/|Z|$ (Burnside).

(d) $G = S_n$: every prime p dividing the degree of an irreducible character of G also divides $n!$. Hence $p \leqslant n$.

(13.7) Theorem (Burnside). If χ is irreducible, then $\chi(1)$ divides $\dfrac{|G|}{|Z|}$.

Proof. Let $\rho : G \to GL(V)$ be a representation with character χ. For any m, consider $\rho_m : \rho^{\otimes m} : G^m \to GL(\bigotimes^m V)$.

Now $\ker \rho_m$ contains the subgroup $Z'_m = \{(g_1, \ldots, g_m) \in Z^m : g_1 \cdots g_m = 1\}$.

If ρ is irreducible then so is ρ_m by (9.11), and $\dim \rho_m = (\dim \rho)^m \, |G^m / Z'_m| = \dfrac{|G|^m}{|Z|^{m-1}}$.

This is true for any m, so $\dim \rho$ divides $\dfrac{|G|}{|Z|}$. (Check via prime factorisation.) $\square$

14. Burnside's $p^a q^b$ Theorem

(14.1) Theorem (Burnside, 1904). p, q primes. Let $|G| = p^a q^b$ where $a, b \in \mathbb{Z}_{\geqslant 0}$, with $a + b \geqslant 2$. Then G is not simple.

Remarks. (1) In fact, even more is true: G is soluble. That is, there exists a chain $G = G^0 \triangleright G^1 \triangleright \cdots \triangleright G^r = \{1\}$ such that G^i/G^{i+1} is abelian for all i.

(2) The result is best possible: A_5 is simple, and $60 = 2^2.3.5$.

(3) If either a or b is 0 then $|G| = p$-power and we know $Z(G) \neq 1$. Then there is $g \in Z$, $|g| = p$ and $\langle g \rangle \lhd G$, with $\langle g \rangle \neq 1$ or G.

(14.2) Proposition. χ an irreducible $\mathbb{C}$-character of G, $\mathcal{C}$ a G-conjugacy class, $g \in G$ such that $(\chi(1), |\mathcal{C}|) = 1$. Then $|\chi(g)| = \chi(1)$ or 0.

Proof. There are $a, b \in \mathbb{Z}_{\geqslant 0}$ such that $a\chi(1) + b|\mathcal{C}| = 1$. Define $\alpha = a\chi(g) + \dfrac{b\chi(g)}{\chi(1)}|\mathcal{C}| = \dfrac{\chi(g)}{\chi(1)}$.

Then α is an algebraic integer, so the assertion follows from:

(14.3) Lemma. Assume $\alpha = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \lambda_i$ is an algebraic integer with $\lambda_j^n = 1$ for all j, some n. Then $|\alpha| = 1$.

For (14.2), we take $n = |g|$, $m = \chi(1)$.

Proof (non-examinable). Assume $|\alpha| \neq 0$. Now $\alpha \in F = \mathbb{Q}(\varepsilon)$ where $\varepsilon = e^{2\pi i/n}$ and $\lambda_j \in F$ for all j.

Let $\mathcal{G} = \mathrm{Gal}(F/\mathbb{Q})$. Observe $\{\beta \in F : \beta^\sigma = \beta \text{ for all } \sigma \in \mathcal{G}\} = F^{\mathcal{G}} = \mathbb{Q}$. (Result from Galois Theory.)

Consider the norm $N(\alpha)$ of α, namely the product of all the Galois conjugates α^σ ($\sigma \in \mathcal{G}$). The norm $\in \mathbb{Q}$ because it's fixed by all of $\mathcal{G}$. It's an algebraic integer (all Galois group conjugates of an algebraic integer are algebraic integers). Hence $N(\alpha) \in \mathbb{Z}$.

But $N(\alpha) = \displaystyle\prod_{\sigma \in \mathcal{G}} \alpha^\sigma$ is a product of expressions $\dfrac{\sum \text{ roots of } 1}{m} \in \mathbb{C}$ of absolute value $\leqslant 1$.

Hence the norm must be ± 1, hence $|\alpha| = 1$. □

(14.4) Theorem. If in a finite group G the number of elements in a conjugacy class $\mathcal{C} \neq \{1\}$ is a p-power, then G is *not* non-abelian simple.

Remark. This implies (14.1). Assume $a > 0, b > 0$. Let $Q \in \mathrm{Syl}_q(G)$. Then $Z(Q) \neq 1$, so choose $1 \neq g \in Z(Q)$. So $C_G(g) \supseteq Q$. Therefore $|\mathcal{C}(g)| = |G : C_G(g)| = p^r$ (some r).

Hence if $p^r = 1$ then $g \in Z(G)$. Therefore $Z(G) \neq 1$ (so not simple). If p^r then G is not simple (by (14.4)).

Proof of (14.4). Assume that G is non-abelian simple, and let $1 \neq g \in G$ with $|\mathcal{C}_G(g)| = p^r$.

By column orthogonality, $0 = \sum_{\substack{\chi \text{ irred} \\ \text{of } G}} \chi(1)\chi(g) \ - (*)$

G is non-abelian simple, so $|\chi(g)| \neq \chi(1)$ for any irreducible $\chi \neq 1$. By (14.2), for any irreducible character $\chi \neq 1$ of G, we have $p|\chi(1)$ or $\chi(g) = 0$.

Deleting zero terms in $(*)$, $0 = 1 + p \sum_{\substack{\chi \text{ irred} \\ p|\chi(1)}} \frac{\chi(1)}{p}\chi(g)$.

Thus $1/p$ is an algebraic integer, since $1/p \in \mathbb{Q}$, hence $1/p \in \mathbb{Z}$. Contradiction. □

Remarks. (a) In 1911, Burnside conjectured that if $|G|$ is odd then G is not non-abelian simple. Only proved in 1963 by Feit & Thompson, a result which began the Classification of Finite Simple Groups. The Classification only ended in 2005.

(b) A group-theoretic proof given only in 1972 (H. Bender)

15. Representations of Topological Groups

(15.1) A **topological group** is a group which is also a topological space such that the group operations $G \times G \to G$, $(h, g) \mapsto hg$ and $G \to G$, $g \mapsto g^{-1}$ are continuous. It is **compact** if it is so as a topological space.

(15.2) Basic examples. (a) $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ are open subspaces of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$.

(b) G finite, discrete topological. Also compact.

(c) $G = S^1 = U(1) = \{g \in \mathbb{C} : |g| = 1\}$.

(d) $O(n) = \{A \in GL_n(\mathbb{R}) : AA^t = I\}$ – orthogonal group.
Compact: set of orthonormal bases for $\mathbb{R}^n = \{(e_1, \ldots, e_n) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n : \langle e_i, e_j \rangle = \delta_{ij}\}$.
$U(n) = \{A \in GL_n(\mathbb{C}) : A\overline{A}^t = I\}$ – unitary group.
Compact: $A \in U(n)$ iff its columns are orthonormal.
Note: $O(1) = \mathbb{Z}/2$, $SO(1) = 1$, and $O(2)/S^1 = \mathbb{Z}/2$, $SO(2) = S^1$.

(e) $SU(n) = \{A \in U(n) : \det A = 1\} = SL_n(\mathbb{C}) \cap U(n)$.

$$\text{E.g.,}\quad SU(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} : z_i \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1 \right\} \text{ – spin group}$$
$$\cong \quad S^3 = \{z \in \mathbb{C}^2 : ||z|| = 1\} \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4 \quad \text{(homeomorphic).}$$

$SO(n) = \{A \in O(n) : \det A = 1\} = SL_n(\mathbb{R}) \cap O(n)$.
E.g., $SO(2) \cong U(1)$, rotation of $\theta \mapsto e^{i\theta}$
$SO(3)$, rotations about various axes in $\mathbb{R}^3$.

$SO(n)$, $SU(n)$, $U(n)$, $O(n)$ are groups of isometries of geometric objects – known as compact **Lie** groups. Theory is done by H. Weyl, 'Classical Groups'.

(15.3) Definition. A **representation** of a topological group on a finite-dimensional vector space V is a continuous group homomorphism $\rho : G \to GL(V)$ with the topology of $GL(V)$ inherited from the space $\operatorname{End} V$.

(There exist extensions when V is infinite-dimensional – see Telemann, remark 19.2.)

Here, continuous $\rho : G \to GL(V) \cong GL_n(\mathbb{C})$ means each $g \mapsto \big(\rho(g)\big)_{ij}$ is continuous for i, j, or the map $G \times V \to V$, $(g, v) \mapsto \rho(g)v$ is continuous.

The compact group $U(1)$

(15.4) Theorem. The continuous homomorphisms $C^1 \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ (i.e. the 1-dim. representations of S^1) are precisely the representations $z \mapsto z^n$ (some $n \in \mathbb{Z}$).

The proof is closely tied with Fourier Series. We need a couple of lemmas.

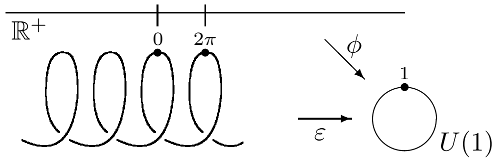
(15.5) Lemma. Consider $(\mathbb{R}, +)$. If $\psi : \mathbb{R} \to \mathbb{R}$ is a continuous homomorphism then ψ is multiplication by a scalar. (I.e., solve $\psi(x + y) = \psi(x) + \psi(y)$ for ψ a continuous function.)

Proof. Put $c = \psi(1)$. Then $\psi(n) = nc$ ($n \in \mathbb{Z}$). Also $m\psi(1/m) = c$, so $\psi(1/m) = c/m$ ($m \in \mathbb{Z}$). Hence $\psi(n/m) = cn/m$. Thus $\psi(x) = cx$ ($x \in \mathbb{Q}$), but $\mathbb{Q}$ is dense in $\mathbb{R}$ and ψ is continuous, so $\psi(x) = cx$ for all $x \in \mathbb{R}$.

(15.6) Lemma. If $\phi : \mathbb{R}^+ \to U(1)$ is a continuous homomorphism then there exists $c \in \mathbb{R}$ with $\phi(x) = e^{icx}$ for all $x \in \mathbb{R}$.

Proof. Claim. There is a unique continuous homomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\phi(x) = e^{i\alpha(x)}$ (so we deduce (15.6) from (15.5)).

Recall that the exponential map $\varepsilon : \mathbb{R}^+ \to U(1)$, $x \mapsto e^{ix}$, maps the real line around the unit circle with period 2π.



complete the triangle!

For any continous $\phi : \mathbb{R}^+ \to U(1)$ such that $\phi(0) = 1$, there exists a unique continuous lifting α of this function to the real line such that $\alpha(0) = 0$ – i.e., there exists a unique continuous $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\phi(x) = \varepsilon(\alpha(x))$ for all x.

(Lifting is constructed starting with condition $\alpha(0) = 0$ and then extending it a small interval at a time. See Telemann, section 21. Non-examinable!)

Claim. If ϕ is a homomorphism then its lift α is also a homomorphism. (So $\alpha(x) = cx$, some c – (15.4).)

We tensor $\phi(a+b) = \phi(a)\phi(b)$, hence $\varepsilon\big(\alpha(a+b) - \alpha(a) - \alpha(b)\big) = 1$. Hence $\alpha(a+b) - \alpha(a) - \alpha(b) = 2\pi m$ for some $m \in \mathbb{Z}$ depending only on a, b. Varying a, b continuously, $m =$ constant; setting $a = b = 0$ shows $m = 0$. $\square$

Proof of (15.4). Given a representation $\rho : S^1 \to \mathbb{C}^\times$, it has a compact, hence bounded, image. This image lies on the unit circle (integral powers of any other complex number would form an unbounded sequence). Thus $\rho : S^1 \to S^1$ is a continuous homomorphism.

Thus we get a homomorphism $\mathbb{R} \to S^1$, $x \mapsto \rho(e^{ix})$, so by (15.6), there exists $c \in \mathbb{R}$ with $\rho(e^{ix}) = e^{icx}$.

Finally, $1 = \rho(e^{i2\pi}) = e^{i2\pi c}$, thus $c \in \mathbb{Z}$. Putting $n = c$ we have $\rho(z) = z^n$. $\square$

So $\rho_n : U(1) \to \mathbb{C}^\times$, $z \mapsto z^n$, $(n \in \mathbb{Z})$ give the complete list of irreducible representations of $U(1)$.

Schur's Lemma applies – all irreducibles are 1-dimensional (cf. (4.4.)). Clearly their characters are linearly independent; in fact they are orthonormal under the inner product

$$\langle \phi, \psi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{\phi(\theta)} \psi(\theta)\, d\theta \qquad (*)$$

where $z = e^{i\theta}$. I.e., 'averaging over $U(1)$'. Finite linear combinations of these ρ_n are the **Fourier polynomials** $= \sum_{m=-n}^{n} a_m \rho_m$; the ρ_n are the **Fourier modes**.

$U(1)$ is abelian, hence coincides with the space of conjugacy classes

(15.7) Theorem. (i) The functions ρ_n form a complete list of the irreducible representations of $U(1)$.

(ii) Every finite-dimensional representation V of $U(1)$ is isomorphic to a sum of the ρ_n. Its character χ_V is a Fourier polynomial. The multiplicity of ρ_n in V equals $\langle \rho_n, \chi_V \rangle$ (as in $(*)$).

Remark. Complete reducibility of a finite-dimensional representation requires invoking Weyl's Unitary Trick (3.4$'$) to average over a given inner product using integration on $U(1)$ – so before moving on to $SU(2)$, let's consider a more general theory of compact groups.

General theory of compact groups

The main tools for studying representations of finite groups are:

- Schur's Lemma – holds here too
- Maschke's Theorem. The relevant proof used Weyl's trick of averaging over G. Need to replace summation by integration over compact group G.

Namely, for each continuous function f on G, we have $\displaystyle\int_G f(g)\,dg \in \mathbb{C}$ such that:

- $\displaystyle\int_G$ is a non-trivial linear functional
- $\displaystyle\int_G$ is left/right-invariant, i.e. $\displaystyle\int_G f(g)\,dg = \int_G f(hg)\,dg = \int_G f(gh)\,dg$ $(h \in G)$
- G has total volume 1, i.e. $\displaystyle\int_G dg = 1$

A (difficult) theorem of Haar asserts that these constraints determine existence and uniqueness for any compact G. We'll assume it, but for our Lie groups of interest ($U(1)$, $SU(2)$, etc) there are easier proofs of existence.

(15.8) Examples. (a) G finite. $\displaystyle\int_G f(g)\,dg = \frac{1}{|G|}\sum_{g\in G} f(g)$.

(b) $G = S^1$. $\displaystyle\int_G f(g)\,dg = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta$.

(c) $G = SU(2)$, 2×2 $\mathbb{C}$-matrices preserving complex inner product and $\det = 1$.
I.e, $SU(2) = \left\{ \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} : |u|^2 + |v|^2 = 1 \right\}$.
Identify G with the unit 3-sphere $S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ in such a way that left/right translation by elements of G give isometries on the sphere. With this identification, translation-invariant integration on G can be taken to be integration over S^3 with usual Euclidean measure $\times 1/2\pi^2$ (to normalise).

(d) Embed $SU(2) \subseteq \mathbb{H} = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} : z_i \in \mathbb{C} \right\}$, the **quaternion algebra**.
(Actually, it's a division algebra, so that every non-zero element has an inverse.)
$\mathbb{H}$ is a 4-dimensional Euclidean space: $||A|| = \sqrt{\det A} = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$, where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, with $SU(2)$ as the unit sphere in this normed space.
Multiplication (from left or right) by an element of $SU(2)$ is an isometry of $\mathbb{H}$, viz:

$$(AX, AX) = \det AX = \det A \, \det X = \det X = (X, X) = (XA, XA),$$

where $A \in SU(2)$.

Once we have found our translation-invariant integration on the set of continuous functions on our compact group G, a lot can be proved about the representation theory of G in parallel with finite groups.

Representations (continuous, finite-dimensional) $\rightsquigarrow$ characters (continuous functions $G \to \mathbb{C}$).

Complete reducibility $\rightsquigarrow$ Weyl's Unitary Trick of averaging over G replaced by integration.

Character inner product: $\langle \chi, \chi' \rangle = \int_G \overline{\chi(g)} \chi'(g)\, dg \quad (\dagger)$

χ irreducible iff $\langle \chi, \chi \rangle = 1$.

Moreover,

(15.9) Theorem. For G compact.

(a) Every finite-dimensional representation is a direct sum of irreducible representations (so completely reducible).

(b) Schur's Lemma applies: if ρ, ρ' are irreducible representations of G then

$$\mathrm{Hom}(\rho, \rho') = \begin{cases} \mathbb{C} & \text{if } \rho \text{ is isomorphic to } \rho' \\ 0 & \text{otherwise} \end{cases}$$

(c) The characters of irreducible representations form an orthonormal set with respect to the inner product $(\dagger)$ above. (The set is infinite, and it is not a basis for the Hilbert space of all continuous class functions.)
Even showing completeness of characters is hard – needs Peter-Weyl Theorem.

(d) If the characters of ρ, ρ' are equal then $\rho \cong \rho'$.

(e) If χ is a character with $\langle \chi, \chi \rangle = 1$ then χ is irreducible.

(f) If G is abelian then all irreducible representations are 1-dimensional.

Note. We don't have actions on finite sets: the regular representation is infinite-dimensional.

Comment. The only spheres with continuous group homomorphisms are S^1 $(= SO(2))$ and the 3-sphere S^3 $(= SU(3))$.

The group $SU(2)$

Recall $G = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$.

$G \to S^3 \hookrightarrow \mathbb{C}^2 = \mathbb{R}^4$, $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mapsto (a_1, a_2, b_1, b_2)$. (Homeomorphism, i.e. continuous inverse.)

The centre is $Z(G) = \{\pm I\}$.

Define the **maximal torus** $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} : |a|^2 = 1 \right\} = S^1$.

Conjugacy

(15.10) Proposition. (a) Every conjugacy class $\mathcal{C}$ of $G = SU(2)$ meets T, i.e. $\mathcal{C} \cap T \neq \emptyset$.

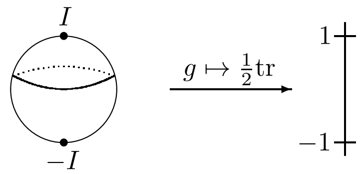
(b) In fact, $\mathcal{C} \cap T = \begin{cases} \{x, x^{-1}\} & \text{if } \mathcal{C} \neq \{\pm I\} \\ \mathcal{C} & \text{if } \mathcal{C} = \{\pm I\} \end{cases}$

(c) The normalised trace, $\frac{1}{2}tr : SU(2) \to \mathbb{C}$, gives a bijection of the set of G-conjugacy classes with the interval $[-1, 1]$, namely

$$g \in \mathcal{C} \mapsto \frac{1}{2}tr = \frac{1}{2}(\lambda + \lambda^{-1}) \text{ if } g \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Picture of ccls:

2-dim spheres of constant latitude on unit sphere, plus the two poles



Choose $x_1 = c$, then $x_2^2 + x_3^2 + x_4^2 = 1 - c^2$, so $-1 < c < 1$.
Given $c \in (-1, 1)$, all matrices $g \in G$ have $\mathrm{tr}\, g = 2c$.

Proof. Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$, $S^2 = -I$.

(a) Every unitary matrix has an orthonormal basis of eigenvectors, hence is conjugate in $U(2)$ to something in T, say $QX\overline{Q}^t \in T$.
We seek Q with $\det Q = 1$ (so that $Q \in SU(2)$).
Let $\delta = \det Q$. Since $Q\overline{Q}^t = I$, $|\delta| = 1$. If ε is a square root of δ then $Q_1 = \overline{\varepsilon} Q \in SU(2)$ (since $\overline{\varepsilon} = 1/\varepsilon$), hence $Q_1 X \overline{Q_1}^t \in T$.

(b) Let $g \in SU(2)$ and suppose $g \in \mathcal{C}_G$. If $g = \pm I$ then $\mathcal{C} \cap T = \{g\}$.
Otherwise g has distinct eigenvalues λ, λ^{-1} and $\mathcal{C} = \left\{ h \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} h^{-1} : h \in G \right\}$.
Thus $\mathcal{C} \cap T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \right\}$, by noting $S \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} S = \begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix}$.
Further, if $\begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix} \in \mathcal{C}$ then $\{\mu, \mu^{-1}\} = \{\lambda, \lambda^{-1}\}$, i.e. the eigenvalues are preserved under conjugacy.

(c) Consider $\frac{1}{2}\mathrm{tr} : \{\text{ccls}\} \to [-1, 1]$. By (b) matrices are conjugate in G iff their eigenvalues agree up to order. Now

$$\frac{1}{2}\mathrm{tr} \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} = \frac{1}{2}(\lambda + \lambda^{-1}) = \mathrm{Re}(\lambda) = \cos\theta \quad (\lambda = e^{i\theta})$$

hence the map is surjective onto $[-1, 1]$.
It's injective: $\frac{1}{2}\mathrm{tr}\,(g) = \frac{1}{2}\mathrm{tr}\,(g')$ then g, g' have the same characteristic polynomial, viz $X^2 - \mathrm{tr}\,(g)X + 1$, hence the same eigenvalues, hence are conjugate. $\square$

Thus we write $\mathcal{C}_t = \{g \in SU(2) : \frac{1}{2}\mathrm{tr}\,(g) = t\}$.

Representations

Let V_n be the space of all homogeneous polynomials of degree n in the variables x, y.

I.e., $V_n = \{r_0 x^n + r_1 x^{n-1}y + \ldots + r_n y^n\}$, an $(n+1)$-dimensional $\mathbb{C}$-space, with basis $x^n, x^{n-1}y, \ldots, y^n$.

(15.11) $GL_2(\mathbb{C})$ acts on V_n.

First, define $\rho_n : GL_2(\mathbb{C}) \to GL(V_n) \cong GL_{n+1}(\mathbb{C})$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\rho_n(g)f(x,y) = f(ax+cy, bx+dy) = f\big((x,y).g\big)$ (i.e., matrix product)

I.e., for $f = \sum_{j=0}^n r_j x^{n-j}y^j$, $\rho(g)f = r_0(ax+cy)^n + r_1(ax+cy)^{n-1}(bx+dy) + \ldots + r_n(bx+dy)^n$.

Check that this defines a representation.

E.g. (a) $n = 0$, $\rho_0 = $ trivial

(b) $n = 1$, natural 2-dimensional representation. $\rho_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the standard basis: $x \mapsto ax + cy$, $y \mapsto bx + dy$.

(c) $n = 2$, $\rho_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has matrix $\begin{pmatrix} a^2 & cb & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$ with respect to the standard basis.
(We have $(ax+cy)^2 + (ax+cy)(bx+dy) + (bx+dy)^2$, so the first column is the coordinate vector of $\rho_2(g)x^2 = (ax+cy)^2 = a^2x^2 + 2acxy + c^2y^2$.)

Characters

$\chi_{V_n}(g) = \mathrm{tr}\,\big(\rho_n(g)\big)$, $g \sim \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \in T$.

$\rho_n \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} x^i y^i = (zx)^i(z^{-1}y)^j = z^{i-j}x^iy^j$.

So $\rho_n \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}$ has matrix $\begin{pmatrix} z^n & & & \\ & z^{n-2} & & \\ & & \ddots & \\ & & & z^{-n} \end{pmatrix}$ with respect to the standard basis.

Hence, $\chi_n = \chi_{V_n} \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} = z^n + z^{n-2} + \ldots + z^{-n}$ $\quad \left[= \dfrac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \quad \text{unless } z = \pm 1.\right]$

(15.12) Theorem. The representations $\rho_n : SU(2) \to GL(V_n)$ of dimension $n+1$ are irreducible for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. Telemann (21.1) shows $\langle \chi_n, \chi_n \rangle = 1$ (implying χ_n irreducible, by (15.9)(e)). We will use combinatorics. Assume $0 \neq W \leqslant V_n$, G-invariant.

Claim. If $w = \sum_j r_j x^{n-j}y^j \in W$ with some $r_j \neq 0$, then $x^{n-j}y^j \in W$.

Proof of claim. We argue by induction on the number of non-zero r_j. If a unique $r_j \neq 0$ then it's clear (multiply by its inverse), so we'll assume more than one and choose one.

Pick $z \in \mathbb{C}$ with $z^n, z^{n-2}, \ldots, z^{-n}$ distinct in $\mathbb{C}$.

Now, $\rho_n \begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} w = \sum r_j z^{n-2j} x^{n-j} y^j \in W$ (G-space).

Define $w_i = \rho_n \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} w - z^{n-2i} w \in W$.

Then $w_i = \sum_j r'_j x^{n-j} y^j$ and $r'_j \neq 0 \Leftrightarrow (r_j \neq 0$ and $j \neq i)$. By induction hypothesis, we have $x^{n-j}y^j \in W$ for all j with ($r_j \neq 0$ and $j \neq i$).

Finally, $x^{n-i}y^i = r_i^{-1}(w - \sum r_j x^{n-j} y^j) \in W$, so the claim is proved.

Now let $0 \neq w \in W$. Wlog, $w = x^{n-j}y^j$. It is now easy to find matrices in $SU(2)$, the action of which will give all the $x^{n-i}y^i \in W$.

E.g.,

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : x^{n-i}y^i \mapsto \frac{1}{\sqrt{2}}(x+y)^{n-i}(-x+y)^i \to x^n \in W$$

$$\underset{(a, b \neq 0)}{\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}} : x^n \mapsto \underset{\substack{\uparrow \\ \text{all coefficients in here } \neq 0}}{(ax+by)^n} \to \text{ all } x^{n-i}y^i \in W$$

So all basis elements are in W. So $W = V_n$. □

Next we show that all irreducibles of $SU(2)$ are of the form in (15.12).

Notation. Write $\mathbb{N}_0[z, z^{-1}] = \left\{ \sum_{m=-n}^{n} a_m z^m : a_m \in \mathbb{N}_0 \right\}$.

And $\mathbb{N}_0[z, z^{-1}]_{\text{ev}} = \{$even Laurent polynomials, i.e. $a_m = a_{-m}$ for all odd $m\}$.

Let $\chi = \chi_V$ be the character of some representation $\rho : G \to GL(V)$. If $g \in G = SU(2)$ then $g \sim_G \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}$ for some $z \in \mathbb{C}$. So χ_V is determined by its restriction to T, hence $\chi_V \in \mathbb{N}_0[z, z^{-1}]$ by (†).

Actually $\chi_V \in \mathbb{N}_0[z, z^{-1}]_{\text{ev}}$, as $\chi_V \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} = \chi_V \begin{pmatrix} z^{-1} & \\ & z \end{pmatrix}$, because $\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \sim_G \begin{pmatrix} z^{-1} & \\ & z \end{pmatrix}$ via $S = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

(15.13) Theorem. Every (finite-dimensional, continuous) irreducible representation of G is one of the $\rho_n : G \to GL(V_n)$ above ($n \geqslant 0$).

Proof. Assume $\rho : G \to GL(V)$ is an irreducible representation affording the character χ. The characters characterise representations (15.9)(d), so it's enough to show $\chi = \chi_n$ for some n.

Now $\chi_0 = 1$, $\chi_1 = z + z^{-1}$, $\chi_2 = z^2 + 1 + z^{-2}$, $\dots$ form a basis of $\mathbb{Q}[z, z^{-1}]_{\mathrm{ev}}$, hence $\chi = \sum a_n \chi_n$, a finite sum with $a_n \in \mathbb{Q}$.

Clear the denominators and move all summands with negative coefficients to the LHS:

$$m\chi + \sum_{i \in I} m_i \chi_i = \sum_{j \in J} n_j \chi_j$$

with I, J disjoint finite subsets of $\mathbb{N}$, and $m, m_i, n_j \in \mathbb{N}$.

The left and right hand sides are characters of representations of $SU(2)$:

$$mV \oplus \bigoplus_I m_i V_i \ \cong \ \bigoplus_J n_j V_j.$$

Since V is irreducible we must have $V \cong V_n$, for some $n \in J$. $\square$

So we have found all irreducible representations of G; they are $\rho_n : G \to GL(V_n)$ $(n \neq 0)$ with V_n the $(n+1)$-dimensional space of homogeneous polynomials of degree n in x, y. The characters of ρ_n are given by (†).

To compute representations we 'just' work with characters: as an example we derive a famous rule for decomposing tensor products.

Tensor product of representations

Recall from section 9: if V, W are G-spaces we have $V \otimes W$ afforidng $\chi_{V \otimes W} = \chi_V \chi_W$.

Examples. $V_1 \otimes V_1 = V_2 \oplus V_0$.

Character $= (z + z^{-1})^2 = z^2 + 2 + z^{-2} = (\underbrace{z^2 + 1 + z^{-2}}_{V_2}) + \underset{V_0}{1}$

$V_1 \otimes V_2 = V_3 \oplus V_1$.

Character $= (z + z^{-1})(z^2 + 1 + z^{-2}) = (z^3 + z + z^{-1} + z^{-3}) + (z + z^{-1})$

(15.14) Theorem (Clebsch-Gordan). $V_n \otimes V_m = V_{n+m} \oplus V_{n+m-2} \oplus \dots \oplus V_{|n-m|}$

Proof. Just check that the characters work.

Wlog $n \geqslant m$ and prove $\chi_n \chi_m = \chi_{n+m} + \chi_{n+m-2} + \dots + \chi_{n-m}$.

$$\begin{aligned} \chi_n(g)\chi_m(g) &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \left(z^m + z^{m-2} + \dots + z^{-m}\right) \\ &= \sum_{j=0}^{m} \frac{z^{n+m+1-2j} - z^{2j-n-m-1}}{z - z^{-1}} \\ &= \sum_{j=0}^{m} \chi_{n+m-2j} \end{aligned}$$

(The $n \geqslant m$ ensures no cancellations in the sum.) $\square$

Some $SU(2)$-related groups

Check (see Telemann 22.1, and Examples Sheet 4 Question 6):

- $SO(3) \cong SU(2)/\{\pm I\}$
- $SO(4) \cong SU(2) \times SU(2)/\{\pm(I, I)\}$ $(*)$
- $U(2) \cong U(1) \times SU(2)/\{\pm(I, I)\}$

(Isomorphisms, but actually homeomorphisms.)

So continuous representations of these groups are the same as continuous representations of $SU(2)$ and $SU(2) \times SU(2)$, respectively, which send $-I$ and $(-I, -I)$ to the identity matrix.

(15.15) Corollary. The irreducible representations of $SO(3)$ are precisely $\rho_{2m} : SO(3) \to GL(V_{2m})$ $(m \geqslant 0)$.

Remarks. (a) We get precisely those V_n with $-\mathrm{id}$ in the kernel of the action, and $-\mathrm{id}$ acts on V_n as

$$\begin{pmatrix} (-1)^n & & & \\ & (-1)^{n-2} & & \\ & & \ddots & \\ & & & (-1)^{-n} \end{pmatrix} = (-1)^n \mathrm{id}$$

(b) V_2 is the standard 3-dimensional representation of $SO(3)$. (The only 3-dimensional representation in the list.)

(c*) For $SO(4)$ the complete list is $\rho_m \otimes \rho_n$ $(m, n \geqslant 0,\ m \equiv n(2))$ (see Telemann 22.7). For $U(2)$ the list is $\det^{\otimes m} \otimes \rho_n$ $(m, n \in \mathbb{Z},\ n \geqslant 0)$ where $\det : U(2) \to U(1)$ is 1-dimensional (see Telemann 22.9).

Sketch proof of $(*)$ Recall from (15.8)(d) that $SU(2) \subseteq \mathbb{H} \cong \mathbb{R}^4$ can be viewed as the space of unit norm quaternions. We also saw that multiplication from the left (and right) by elements of $SU(2)$ gives isometries of $\mathbb{H}$. The left/right multiplication action of $SU(2)$ gives a homomorphism $\phi : SU(2) \times SU(2) \to SO(4)$, $(g, h) \mapsto \{\theta : q \mapsto gqh^{-1}\}$.

Kernel. (g, h) sends $1 \in \mathbb{H}$ to gh^{-1}, so (g, h) fixes the identity iff $g = h$, i.e. $G = \{(g, g) : g \in SU(2)\} = \mathrm{stab}_{SU(2)\times SU(2)}(1)$.

Now (g, g) fixes every other quaternion iff $g \in Z(SU(2))$, i.e. $g = \pm\mathrm{id}$. Thus $\ker\phi = \{\pm(I, I)\}$.

Surjective and homeomorphic (i.e. inverse map is continuous). Restricting the left/right action to G (the diagonal embedding of $SU(2)$) give the conjugation action of $SU(2)$ on the space of 'pure quaternions', $\langle \underline{i}, \underline{j}, \underline{k} \rangle_{\mathbb{R}}$ (the trace 0 skew-Hermitian 2×2 matrices). So get a 3-dimensional Euclidean space on which G acts, and $\phi(G) \leqslant SO(3)$.

$\phi(G) = SO(3)$**.** Rotations in $(\underline{i}, \underline{j})$-plane implemented by $a + b\underline{k}$, similarly with any permutations of $\underline{i}, \underline{j}, \underline{k}$, and these rotations generate $SO(3)$ (see some Geometry course). So we have a surjective homomorphism $SU(2) \to SO(3)$, and we know that $\ker = \{\pm\mathrm{id}\}$. The result follows.

Homeomorphism. Prove it directly or 'recall' the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism (Sutherland 5.9.1)

Further worked example

S_n, $GL_2(\mathbb{F}_q)$, H_p.

We consider Heisenberg groups. For p prime, the abelian groups of order p^3 are C_{p^3}, $C_{p^2} \times C_p$, $C_p \times C_p \times C_p$, and their character tables can be constructed using (4.5).

Suppose G is any non-abelian group of order p^3. Let $Z = Z(G)$, then it's well-known that $Z \neq 1$ and G/Z is non-cyclic, i.e. $G/Z \cong C_p \times C_p$ and $Z = C_p$.

Take $G = H_p = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_p \right\}$, the **modular Heisenberg group**.

We take p odd (else $G = D_8$ or Q_8).

Have $Z = \langle z \rangle$, $z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

With $a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $[a, b] = z$ and $G' = Z$.

There are p^2 linear characters (of degree 1) (recall $G/G' = C_p \times C_p$), and $(p-1)$ characters of degree p, induced from the 1-dimensional characters of the abelian subgroup

$$\langle a, z \rangle = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

of order p^2.

Conjugacy classes

p conjugacy classes of size 1. The rest have size p and there are $p^2 - 1$ such classes.

We'll show that the character table of H_p looks like

	$\longleftarrow$ p central ccls $\longrightarrow$				$\leftarrow p^2-1$ ccls each of size $p \rightarrow$			
	1	z	$\dots$	z^{p-1}	a	ab	$\dots$	$a^{-1}b^{-1}$
p^2 linear characters	1	1	$\dots$	1	char. table of $C_p \times C_p$ lifted			
	1	1	$\dots$	1				
	$\vdots$	$\vdots$	$\ddots$	$\vdots$				
	1	1	$\dots$	1				
$p-1$ characters of degree p	p	$p\omega$	$\dots$	$\dots$	0			
	p	$p \times$ char. table of C_p						
	$\vdots$							
	p							

More formally,

- $Z = \langle z \rangle$ gives p conjugacy classes of size 1: $\{1\}$, $\{z\}$, …, $\{z^{p-1}\}$.
- $G/Z = \langle aZ, bZ \rangle = \{a^i b^j Z : 0 \leqslant i \leqslant p-1, 0 \leqslant j \leqslant p-1\}$.
  So, in particular, every element of G is of the form $a^i b^j z^k$, $0 \leqslant i, j, k, \leqslant p-1$.
- the p^2-1 conjugacy classes of size p are $\mathcal{C}(a^i b^j) = \{a^i b^j z^k : 0 \leqslant k \leqslant p-1, (i,j) \neq (0,0)\}$.
  For $aba^{-1}b^{-1} = z$: $\begin{array}{l} aba^{-1} = zb \quad (= bz \text{ as } z \text{ central}) \\ bab^{-1} = az^{-1} \end{array}$

  $\Rightarrow \begin{array}{l} aa^ib^ja^{-1} = a^i(aba^{-1})^j = a^ib^jz^j \\ ba^ib^jb^{-1} = (bab^{-1})^ib^j = a^ib^jz^{-i} \end{array}$.

  I.e., any conjugate of a^ib^j is some $a^ib^jz^k$, as above.

Irreducible characters

(15.16) Theorem. As above, let $G = \{a^i b^j z^k : 0 \leqslant i, j, k \leqslant p-1\}$ be a non-abelian group of order p^3. Write $\omega = e^{2\pi i/p} \in \mu_p$. Then the irreducible characters of G are:

$$\begin{array}{ccc} \chi_{u,v} & (0 \leqslant u, v \leqslant p-1) & (p^2 \text{ of degree } 1) \\ \phi_u & (1 \leqslant u \leqslant p-1) & (p^2 - 1 \text{ of degree } p \end{array}$$

where for all i, j, k,

$$\begin{array}{rcl} \chi_{u,v}(a^ib^jz^k) & = & w^{iu+jv} \\ \phi_u(a^ib^jz^k) & = & \left\{ \begin{array}{cl} p\omega^{uk} & \text{if } i = j = 0 \\ 0 & \text{otherwise} \end{array} \right. \end{array}$$

Proof. First, the p^2 linear characters. The irreducible characters of $G^{\mathrm{ab}} = G/G' = G/Z = C_p \times C_p$ are $\psi_{u,v}(a^ib^jZ) = \omega^{iu+jv}$ $(0 \leqslant u, v \leqslant p-1)$. The lift to G of $\psi_{u,v}$ is precisely $\chi_{u,v}$.

Next, the $p-1$ characters of degree p.

Now, $H = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_p \right\} \cong \langle a, z \rangle$ is a normal abelian subgroup of index p.

Let ψ_u be a character of H defined as $\psi_u(a^iz^k) = \omega^{uk}$ $(0 \leqslant k \leqslant p-1)$, and calculate ψ_u^G. Choose transversal $\{1, b, \ldots, b^{p-1}\}$ of H in G.

$$\begin{array}{rcl} \psi_u^G(a^iz^k) & = & \psi_u(a^i) + \psi_u(a^iz) + \ldots + \psi_u(a^iz^{p-1}) \\ & = & \psi_u(a^i) \displaystyle\sum_{r=0}^{p-1} \psi_u(z^r) \quad \text{(as homomorphic)} \\ & = & \psi_u(a^i) \displaystyle\sum_{r=0}^{p-1} \omega^{ur} \;=\; 0 \end{array}$$

$\psi_u^G(z^k) = \displaystyle\sum_j \mathring{\psi}_u(b^jz^kb^{-j}) = p\,\psi_u(z^k) = p\omega^{uk}$, and $\psi_u(g) = 0$ for all $g \notin H$.

Thus $\psi_u^G = \phi_u$.

Finally,

$$\begin{aligned}\langle \phi_n, \phi_n \rangle &= \frac{1}{p^3} \sum_{g \in G} \overline{\phi_u(g)} \phi_u(g) \;=\; \frac{1}{p^3} \sum_{g \in Z} \overline{\phi_u(g)} \phi_u(g) \\ &= \frac{1}{p^3} \sum_{k=0}^{p-1} \overline{\phi_u(z^k)} \phi_u(z^k) \;=\; \frac{1}{p^3} \sum_{z} p^2 \;=\; 1\end{aligned}$$

□

Remarks. 1. Alternative is to apply Mackey (12.6).

2. Typically for p-groups: any irreducible representation is induced from a 1-dimensional representation of some subgroup (Telemann, chapter 17).

3. For p odd, in fact there are two non-abelian groups of order p^3:

$$G_1 = \langle a, b : a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle \text{ with } Z = \langle a^p \rangle$$
$$G_2 = \langle a, b, z : a^p = b^p = z^p = 1, az = za, bz = zb, b^{-1}ab = az \rangle \text{ with } Z = \langle z \rangle$$

PART II REPRESENTATION THEORY
SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let ρ be a representation of the group G.
(a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G.
(b) Prove that $G/\ker\delta$ is abelian.
(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, and let $\rho : G \to \mathrm{GL}(V)$ be another representation. Show that $\theta\otimes\rho : G \to \mathrm{GL}(V)$ given by $\theta\otimes\rho : g \mapsto \theta(g)\cdot\rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

3 Let G be the alternating group A_4. Find all the degree one representations of G over F for:
(a) $F = \mathbb{C}$;
(b) $F = \mathbb{R}$;
(c) $F = \mathbb{Z}/3\mathbb{Z}$.
[Hint: you can use the fact that the Klein 4-group $V = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is the unique normal subgroup of A_4 (apart from the trivial subgroup and A_4 itself).]
Now let $G = D_{12}$, the symmetry group of a regular hexagon. Let $a \in G$ be a rotation through $\pi/3$ anticlockwise, and let $b \in G$ be a reflection, so that $G = \{a^i, a^ib \ : \ 0 \leqslant i \leqslant 5\}$. Let $A, B, C, D \in \mathrm{GL}_2(\mathbb{C})$ be the matrices

$$A = \begin{pmatrix} e^{\pi i/3} & 0 \\ 0 & e^{-\pi i/3} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Each of the following is a (matrix) representation of G over $\mathbb{C}$ (you need not verify this):
$\rho_1 : a^rb^s \mapsto A^rB^s$.
$\rho_2 : a^rb^s \mapsto A^{3r}(-B)^s$.
$\rho_3 : a^rb^s \mapsto (-A)^rB^s$.
$\rho_4 : a^rb^s \mapsto C^rD^s$.
Which of these are faithful? Which are equivalent to one another?

4 (Counterexamples to Maschke's Theorem)
(a) Let FG denote the regular FG-module (i.e. the permutation module coming from the action of G on itself by left multiplication), and let F be the trivial module. Find all the FG-homomorphisms from FG to F and vice versa. By considering a submodule of FG isomorphic to F, prove that whenever the characteristic of F divides the order of G, there is a counterexample to Maschke's Theorem.
(b) Find an example of a representation of some finite group over some field of characteristic p, which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

5 Let N be a normal subgroup of the group G. Given a representation of the quotient G/N, use it to obtain a representation of G. Which representations of G do you get this way?

Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from G/G'.

6 Let G be a cyclic group of order n. Decompose the regular representation of G explicitly as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g\in G}$ to a basis where the group action is diagonal.

7 Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y \ : \ x^5 = 1 = y^2, \ yxy^{-1} = x^{-1}\rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbb{R}$.

8 Let G be the quaternion group Q_8 of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, \ yxy^{-1} = x^{-1}\rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

9 State Schur's lemma.

Show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

10 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56)\rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

11 In this question work over the field $F = \mathbb{R}$.

Let G be the cyclic group of order 3.

(a) Write the regular $\mathbb{R}G$-module as a direct sum of irreducible submodules.

(b) Find all the $\mathbb{R}G$-homomorphisms between the irreducible $\mathbb{R}G$-modules.

(c) Show that the conclusion of Schur's Lemma ('every homomorphism from an irreducible module to itself is a scalar multiple of the identity') is false if you replace $\mathbb{C}$ by $\mathbb{R}$.

From now on let G be a cyclic group of order n. Show that:

(d) If n is even, the regular $\mathbb{R}G$-module is a direct sum of two (non-isomorphic) 1-dimensional irreducible submodules and $(n-2)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

(e) If n is odd, the regular $\mathbb{R}G$-module is a direct sum of one 1-dimensional irreducible submodule and $(n-1)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

[Hint: use the fact that $\mathbb{R}G \subset \mathbb{C}G$ and what you know about the regular $\mathbb{C}G$-module from question 6.]

12 Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$.

SM, Lent Term 2011

Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`

PART II REPRESENTATION THEORY
SHEET 2

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G of dimension d, and affording character χ. Show that $\ker \rho = \{g \in G \mid \chi(g) = d\}$. Show further that $|\chi(g)| \leqslant d$ for all $g \in G$, with equality only if $\rho(g) = \lambda I$, a scalar multiple of the identity, for some root of unity λ.

2 Let χ be the character of a representation V of G and let g be an element of G. If g is an involution (i.e. $g^2 = 1 \neq g$), show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$. If G is simple (but not C_2), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. (Hint: consider the determinant of g acting on V.) If g has order 3 and is conjugate to g^{-1}, show that $\chi(g) \equiv \chi(1) \bmod 3$.

3 Construct the character table of the dihedral group D_8 and of the quaternion group Q_8. You should notice something interesting.

4 Construct the character table of the dihedral group D_{10}.

Each irreducible representation of D_{10} may be regarded as a representation of the cyclic subgroup C_5. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of C_5.

Repeat for $D_{12} \cong S_3 \times C_2$ and the cyclic subgroup C_6 of D_{12}.

5 Construct the character tables of A_4, S_4, S_5, and A_5.

The group S_n acts by conjugation on the set of elements of A_n. This induces an action on the set of conjugacy classes and on the set of irreducible characters of A_n. Describe the actions in the cases where $n = 4$ and $n = 5$.

6 The group M_9 is a certain subgroup of the symmetric group S_9 generated by the two elements $(1,4,9,8)(2,5,3,6)$ and $(1,6,5,2)(3,7,9,8)$. You are given the following facts about M_9:

• there are six conjugacy classes:

— C_1 contains the identity.

— For $2 \leqslant i \leqslant 4$, $|C_i| = 18$ and C_i contains g_i, where $g_2 = (2,3,8,6)(4,7,5,9)$, $g_3 = (2,4,8,5)(3,9,6,7)$ and $g_4 = (2,7,8,9)(3,4,6,5)$.

— $|C_5| = 9$, and C_5 contains $g_5 = (2,8)(3,6)(4,5)(7,9)$

— $|C_6| = 8$, and C_6 contains $g_6 = (1,2,8)(3,9,4)(5,7,6)$.

• every element of M_9 is conjugate to its inverse.

Calculate the character table of M_9. [Hint: You may find it helpful to notice that $g_2^2 = g_3^2 = g_4^2 = g_5$.]

7 A certain group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β. The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

8 The table below is a part of the character table of a certain finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

	1	3	3	7	7
χ_1	1	1	1	ζ	$\bar{\zeta}$
χ_2	3	γ	$\bar{\gamma}$	0	0
χ_3	3	$\bar{\gamma}$	γ	0	0

9 Let x be an element of order n in a finite group G. Say, without detailed proof, why
(a) if χ is a character of G, then $\chi(x)$ is a sum of nth roots of unity;
(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};
(c) x and x^{-1} have the same number of conjugates in G.

Prove that the number of irreducible characters of G which take only real values (so-called *real characters*) is equal to the number of self-inverse conjugacy classes (so-called *real classes*).

A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.
[You may assume, if needed, the fact that $\sqrt{7}$ is not in the field $\mathbb{Q}(\zeta)$, where ζ is a primitive 7th root of unity.]

10 Let a finite group G act on itself by conjugation. Find the character of the corresponding permutation representation.

11 Consider the character table Z of G as a matrix of complex numbers (as we did when deriving the column orthogonality relations from the row orthogonality relations).

(a) Using the fact that the complex conjugate of an irreducible character is also an irreducible character, show that the determinant $\det Z$ is $\pm \det \bar{Z}$, where $\bar{Z}$ is the complex conjugate of Z.

(b) Deduce that either $\det Z \in \mathbb{R}$ or $i.\det Z \in \mathbb{R}$.

(c) Use the column orthogonality relations to calculate the product $\bar{Z}^T Z$, where $\bar{Z}^T$ is the transpose of the complex conjugate of Z.

(d) Calculate $|\det Z|$.

12 The character table obtained in Question 9 is in fact the character table of the group $G = \mathrm{PSL}_2(7)$ of 2×2 matrices with determinant 1 over the field $\mathbb{F}_7$ (of seven elements) modulo the two scalar matrices.

Deduce directly from the character table which you have obtained that G is simple.

[Comment: it is known that there are precisely five non-abelian simple groups of order less than 1000. The smallest of these is $A_5 \cong \mathrm{PSL}_2(5)$, while G is the second smallest. It is also known that for $p \geqslant 5$, $\mathrm{PSL}_2(p)$ is simple.]

Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space $(\mathbb{F}_7)^2$). Obtain the permutation character of this action and decompose it into irreducible characters.

Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.

[Hint: for the last part use the formula that the number of pairs of elements conjugate to x and y respectively, whose product is conjugate to t, equals $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$, where the sum runs over all the irreducible characters of G, and $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]

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PART II REPRESENTATION THEORY
SHEET 3

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Recall the character table of S_4 from Sheet 2. Find all the characters of S_5 induced from the irreducible characters of S_4. Hence find the complete character table of S_5.

Repeat, replacing S_4 by the subgroup $\langle(12345),(2354)\rangle$ of order 20 in S_5.

2 Recall the construction of the character table of the dihedral group D_{10} of order 10 from Sheet 2.

(a) Use induction from the subgroup D_{10} of A_5 to A_5 to obtain the character table of A_5.

(b) Let G be the subgroup of $\mathrm{SL}_2(\mathbb{F}_5)$ consisting of upper triangular matrices. Compute the character table of G.

Hint: bear in mind that there is an isomorphism $G/Z \to D_{10}$.

3 Let H be a subgroup of the group G. Show that for every irreducible representation ρ for G there is an irreducible representation ρ' for H with ρ a component of the induced representation $\mathrm{Ind}_H^G\,\rho'$.

Prove that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G:A|$.

4 Obtain the character table of the dihedral group D_{2m} of order $2m$, by using induction from the cyclic subgroup C_m. [Hint: consider the cases m odd and m even separately, as for m even there are two conjugacy classes of reflections, whereas for m odd there is only one.]

5 Prove the transitivity of induction: if $H < K < G$ then

$$\mathrm{Ind}_K^G\,\mathrm{Ind}_H^K\,\rho \cong \mathrm{Ind}_H^G\rho$$

for any representation ρ of H.

6

(a) Let $V = U \oplus W$ be a direct sum of $\mathbb{C}G$-modules. Prove that both the symmetric square and the exterior square of V have submodules isomorphic to $U \otimes W$.

(b) Calculate $\chi_{\Lambda^2\rho}$ and $\chi_{S^2\rho}$, where ρ is the irreducible representation of dimension 2 of D_8; repeat this for Q_8. Which of these characters contains the trivial character in the two cases?

7 Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G of dimension d.

(a) Compute the dimension of S^nV and Λ^nV for all n.

(b) Let $g \in G$ and let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of g on V. What are the eigenvalues of g on S^nV and Λ^nV?

(c) Let $f(x) = \det(g - xI)$ be the characteristic polynomial of g on V. Describe how to obtain the trace $\chi_{\Lambda^nV}(g)$ from the coefficients of $f(x)$.

(d)* Find a relation between $\chi_{S^nV}(g)$ and the polynomial $f(x)$. [Hint: first do the case when $\dim V = 1$.]

8 Let G be the symmetric group S_n acting naturally on the set $X = \{1, \dots, n\}$. For any integer $r \leqslant \frac{n}{2}$, write X_r for the set of all r-element subsets of X, and let π_r be the permutation character of the action of G on X_r. Observe $\pi_r(1) = |X_r| = \binom{n}{r}$. If $0 \leqslant \ell \leqslant k \leqslant n/2$, show that

$$\langle \pi_k, \pi_\ell \rangle = \ell + 1.$$

Let $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd. Deduce that S_n has distinct irreducible characters $\chi^{(n)} = 1_G$, $\chi^{(n-1,1)}, \chi^{(n-2,2)}, \dots, \chi^{(n-m,m)}$ such that for all $r \leqslant m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \cdots + \chi^{(n-r,r)}.$$

In particular the class functions $\pi_r - \pi_{r-1}$ are irreducible characters of S_n for $1 \leqslant r \leqslant n/2$ and equal to $\chi^{(n-r,r)}$.

9 If $\rho : G \to \mathrm{GL}(V)$ is an irreducible complex representation for G affording character χ, find the characters of the representation spaces $V \otimes V$, $\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$.

Define the *Frobenius-Schur indicator* $\iota\chi$ of χ by

$$\iota\chi = \frac{1}{|G|} \sum_{x \in G} \chi(x^2)$$

and show that

$$\iota\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

[Remark. The sign +, resp. −, indicates whether $\rho(G)$ preserves an orthogonal, respectively, symplectic form on V, and whether or not the representation can be realised over the reals. You can read about it in Isaacs or in James and Liebeck.]

10 If θ is a faithful character of the group G, which takes r distinct values on G, prove that each irreducible character of G is a constituent of θ to power i for some $i < r$.
[Hint: assume that $\langle \chi, \theta^i \rangle = 0$ for all $i < r$; use the fact that the Vandermonde $r \times r$ matrix involving the row of the distinct values $a_1, \dots, a_r$ of θ is nonsingular to obtain a contradiction.]

11 Construct the character table of the symmetric group S_6. Identify which of your characters are equal to the characters $\chi^{(6)}, \chi^{(5,1)}, \chi^{(4,2)}, \chi^{(3,3)}$ constructed in question 8.

12 Show by induction on n that if the symmetric group S_n with $n \geq 5$ has a complex irreducible representation ρ of dimension $d \leq n$ then one of the following holds:

(i) $d = 1$, and ρ is either the trivial representation 1 or the sign representation σ;

(ii) $d = n - 1$ and either $1 \oplus \rho$ or $1 \oplus \sigma\rho$ is the natural permutation representation;

(iii) $n = 5$ and $d = 5 (= n)$ or $n = 6$ and $d = 5 (= n - 1)$.

[Hint: Restrict ρ to $S_{n-2} \times S_2$; it becomes reducible, unless it is linear; now use induction - what do linear representations of $S_{n-2} \times S_2$ get induced to?]

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PART II REPRESENTATION THEORY
SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over $\mathbb{C}$. In the first nine questions we let $G = \mathrm{SU}(2)$.

1 (a) Let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. Describe the character χ_n of ρ_n.

(b) Decompose $V_4 \otimes V_3$ into irreducible G-spaces (that is, find a direct sum of irreducible representations which is isomorphic to $V_4 \otimes V_3$. In this and the following questions, you are not being asked to find such an isomorphism explicitly.)

(c) Decompose also $V_3^{\otimes 2}$, $\Lambda^2 V_3$ and $S^2 V_3$.

(d) Show that V_n is isomorphic to its dual V_n^*.

2 Decompose $V_1^{\otimes n}$ into irreducibles.

3 Determine the character of $S^n V_1$ for $n \geq 1$.
Decompose $S^2 V_n$ and $\Lambda^2 V_n$ for $n \geq 1$.
Decompose $S^3 V_2$ into irreducibles.

4 Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices, by

$$A : \ X \ \mapsto \ A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducibles.

5 Let χ_n be the character of the irreducible representation ρ_n of G on V_n.
Show that

$$\frac{1}{2\pi} \int_0^{2\pi} K(z) \chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi} \int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

6 (a) Let G be a compact group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive definite) symmetric bilinear form.
(b) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbb{R}^3$, that $(A, B) \mapsto -\mathrm{tr}(AB)$ is a positive definite non-degenerate invariant bilinear form, and that ρ is surjective with kernel $\{\pm I\}$.

7 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines an G-invariant inner product on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

8 Suppose we are given that H is a subgroup of order 24 in G. We are told that H contains $\{\pm I\}$ as a normal subgroup, and that the quotient group $H/\{\pm I\}$ is isomorphic to A_4. Find the character table of H. [You may assume that H has a conjugacy class containing six elements of order 4, two conjugacy classes each containing four elements of order 3, and two conjugacy classes each containing four elements of order 6.]

9 Compute the character of the representation $S^n V_2$ of G for any $n \geqslant 0$. Calculate $\dim_{\mathbb{C}}(S^n V_2)^G$ (by which we mean the subspace of $S^n V_2$ where G acts trivially).

Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of SO(3) is a polynomial ring. Find a generator for this polynomial ring.

10 The *Heisenberg group* of order p^3 is the (non-abelian) group

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of 3×3 upper unitriangular matrices over the finite field $\mathbb{F}_p$ of p elements (p prime).

Show that G has p conjugacy classes of size 1, and $p^2 - 1$ conjugacy classes of size p.

Find p^2 characters of degree 1.

Let H be the subgroup of G comprising matrices with $a = 0$. Let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial 1-dimensional representation of the cyclic group $\mathbb{F}_p = \mathbb{Z}/p$, and define a 1-dimensional representation ρ of H by

$$\rho \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Check that $V_\psi = \mathrm{Ind}_H^G \rho$ is irreducible.

Now list all the irreducible representations of G, explaining why your list is complete.

11 Recall Sheet 3, q.8 where we used inner products to construct some irreducible characters $\chi^{(n-r,r)}$ for S_n. Let $n \in \mathbb{N}$, and let Ω be the set of all ordered pairs (i, j) with $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Let $G = S_n$ act on Ω in the obvious manner (namely, $\sigma(i, j) = (\sigma i, \sigma j)$ for $\sigma \in S_n$). Let's write $\pi^{(n-2,1,1)}$ for the permutation character of S_n in this action.

Prove that

$$\pi^{(n-2,1,1)} = 1 + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \psi,$$

where ψ is an irreducible character. Writing $\psi = \chi^{(n-2,1,1)}$, calculate the degree of $\chi^{(n-2,1,1)}$. Find its value on any transposition and on any 3-cycle. Returning to the character table of S_6 calculated on Sheet 3 q.11, identify the character $\chi^{(4,1,1)}$.

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