Algebraic Topology

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Lent Term 2011

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Examples Sheets

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ALGEBRAIC TOPOLOGY (D) **24 lectures, Lent term**

Either Analysis II or Metric and Topological Spaces is essential.

The fundamental group

Homotopy of continuous functions and homotopy equivalence between topological spaces. The fundamental group of a space, homomorphisms induced by maps of spaces, change of base point, invariance under homotopy equivalence. [3]

The fundamental group of the circle

Covering spaces and covering maps. Path-lifting and homotopy-lifting properties, and the fundamental group of the circle. Topological proof of the fundamental theorem of algebra. [3]

Computing the fundamental group

Free groups, generators and relations for groups. Van Kampen's theorem. The fundamental group of the n-sphere, and of the closed (orientable) surface of genus g. [3]

Covering spaces

Statement of the correspondence, for a connected, locally contractible space X, between coverings of X and conjugacy classes of subgroups of the fundamental group of X, *and its proof*. The universal covering of X. The fundamental group of tori and real projective spaces. [3]

Simplicial complexes

Finite simplicial complexes and subdivisions; the simplicial approximation theorem *and its proof*. [2]

Homology

Simplicial homology, the homology groups of a simplex and its boundary. Functorial properties; the invariance of simplicial homology groups under homotopy equivalence. The homology of S^n; Brouwer's fixed-point theorem. [6]

Homology calculations

The Mayer-Vietoris sequence. Determination of the homology groups of closed surfaces. Orientability for surfaces. *The fundamental group and the first homology group.* The Euler characteristic. *Sketch of the classification of closed triangulable surfaces.* [4]

Appropriate books

M. A. Armstrong *Basic topology.* Springer 1983 (£38.50 hardback)
W. Massey *A basic course in algebraic topology.* Springer 1991 (£50.00 hardback)
C. R. F. Maunder *Algebraic Topology.* Dover Publications 1980 (£11.95 paperback)
A. Hatcher *Algebraic Topology.* Cambridge University Press, 2001 (£20.99 paperback)

Preamble: why algebraic topology?

Problem A

Topological invariance of dimension

If $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic, do we necessarily have $m = n$?

Compare: if $\mathbb{R}^n \cong \mathbb{R}^m$ as vector spaces, is $m = n$? Yes.

We can represent linear maps $\mathbb{R}^n \to \mathbb{R}^m$ by $m \times n$ matrices: any invertible matrix is square (combinatorial fact).

Fact 1. There exist continuous bijective mappings $\mathbb{R}^1 \to \mathbb{R}^2$. (The inverse is not continuous, so this is not a homeomorphism.)†

Fact 2. $\mathbb{Q}^n$ and $\mathbb{Q}^m$ are homeomorphic for all m, n.

Problem B

Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, and consider $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, $f(x) = \frac{x}{\|x\|}$.

This f is a **retraction**, i.e. $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$.

Is there a continuous retraction $g : \mathbb{R}^n \to S^{n-1}$?

Solution: turn topology into algebra.

We construct **functors** from topological spaces to groups (think 'algebraic structures'), i.e. operators F assigning to any space X a group FX, and to any continuous map $f : X \to Y$ a homomorphism $Ff = f_* : FX \to FY$ in such a way that given $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have $(gf)_* = g_* f_* : FX \to FZ$, and $(\mathrm{id}_X)_* = \mathrm{id}_{FX}$.

Given such an F, we see that if X and Y are homeomorphic then FX and FY are isomorphic.

Similarly, if $X' \subset X$ and there is a retraction $g : X \to X'$, then, writing $i : X' \hookrightarrow X$ for inclusion, since $gi = \mathrm{id}_{X'}$, we must have $g_* i_* = \mathrm{id}_{FX'}$. So if we can find F such that Fi isn't injective then no such g exists.

†This isn't quite true. It is nearly true, in that there are continuous surjective mappings $\mathbb{R}^1 \to \mathbb{R}^2$ which are injective except on a small set. Exercise: there is no continuous bijection $\mathbb{R}^1 \to \mathbb{R}^2$.

Chapter 1 : Homotopy and the Fundamental Group

1.1 Definitions.

(a) Suppose we have $f, g : X \to Y$. By a **homotopy** from f to g, we mean a continuous map $H : X \times I \to Y$ (where I is the cloesd unit interval $[0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

(b) We say that f and g are **homotopic**, and write $f \simeq g$, if there exists a homotopy from f to g.

(c) We say that X and Y are **homotopy equivalent**, and write $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. (This is a weakening of 'homeomorphic'.)

1.2 Lemma.

(i) Homotopy gives an equivalence relation on the set of continuous maps $X \to Y$.

(ii) Given $X \xrightarrow{k} Y \xrightarrow{f,g} Z \xrightarrow{h} W$, if $f \simeq g$ then $hf \simeq hg$ and $fk \simeq gk$.

(iii) Homotopy equivalence is an equivalence relation on topological spaces.

Proof.

(i) $f \simeq f$ by the homotopy $H(x,t) = f(x)$ for all t.
If $f \simeq g$ by H, then $g \simeq f$ by $\overline{H}$, where $\overline{H}(x,t) = H(x, 1-t)$.
If $f \simeq g$ by H and $g \simeq h$ by K, then $f \simeq h$ by $H \cdot K$, where

$$H \cdot K(x,t) = \begin{cases} H(x, 2t) & (0 \leqslant t \leqslant \frac{1}{2}) \\ K(x, 2t-1) & (\frac{1}{2} \leqslant t \leqslant 1) \end{cases}$$

(Check that this is well-defined.)

(ii) Suppose $H : Y \times I \to Z$ is a homotopy from f to g. Then $hH : Y \times I \to W$ is a homotopy from hf to hg, and $H(k \times \mathrm{id}_I) : X \times I \to Z$ is a homotopy from fk to gk.

(iii) Reflexivity and symmetry are obvious.

Given homotopy equivalences $X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{k}{\overset{h}{\rightleftarrows}} Z$, we have $(gk)(hf) = g(kh)f \simeq g(\mathrm{id}_Y)f = gf \simeq \mathrm{id}_X$, and $(hf)(gk) \simeq \mathrm{id}_Z$ similarly. □

We say that a space X is **contractible** if $X \simeq \{x\}$; equivalently, if id_X is homotopic to a constant map $X \to X$.

If $A \subset X$ and $r : X \to X$ is such that $r(X) \subset A$, $r \simeq \mathrm{id}_X$ and $r|_A = \mathrm{id}_A$, then r is called a **deformation retraction** of X to A. I.e., $r : X \to A$, $i : A \hookrightarrow X$, $ri = \mathrm{id}_A$, $ir = r \simeq \mathrm{id}_X$, so X and A are homotopy equivalent.

1.3 Examples.

(i) $\mathbb{R}^n$ is contractible.
We have a homotopy $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ defined by $H(x,t) = tx$. Clearly $H(x,0) = 0$ and $H(x,1) = x$ for all x.

(ii) $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$.
We have $i : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, where $x \mapsto \frac{x}{\|x\|}$. Then $fi = \mathrm{id}_{S^{n-1}}$, and we have a homotopy from $\mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ to if given by $H(x,y) = (1-t)x + t\frac{x}{\|x\|}$. (Such a homotopy is called a **linear homotopy**.)

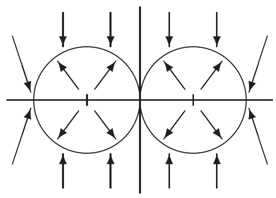
(iii) If $X \subset \mathbb{R}^n$ is a convex subset (i.e. if $x, y \in X$ then $tx + (1-t)y \in X$ for all $0 \leqslant t \leqslant 1$), then any two maps $Y \to X$ are homotopic.

(iv) Let x, y be two points in $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus \{x, y\} \simeq S^{n-1} \vee S^{n-1}$, the **wedge union** of two copies of S^{n-1}.

Given X, Y, define $X \vee Y$ to be the quotient of the disjoint union of X and Y by the equivalence relation which identifies a single point $x \in X$ with a single point $y \in Y$. (Because spheres are uniform, the ambiguity of notation doesn't matter here.)

We claim that the inclusion i of the two circles in $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ is a homotopy equivalence.

We define $f : \mathbb{R}^2 \setminus \{(\pm 1, 0)\} \to S^1 \vee S^1$ separately on the interiors of the two circles, on the region outside the circles with $|x| \leqslant 2$, and on the regions with $|x| \geqslant 2$, as shown in the picture.



Then $if \simeq \mathrm{id}$ by a linear homotopy.

1.4 Definition. Let X be a topological space.

(a) By a **path** in X from x to y, we mean a continuous map $u : I \to X$ with $u(0) = x$ and $u(1) = y$.

(b) By a **loop** in X based at x, we mean a path from x to x. Equivalently, a path is a homotopy between two maps $\{x\} \to X$.

Given a path u from x to y and a path v from y to z, we may define the **product path** $u \cdot v$ from x to z, given by

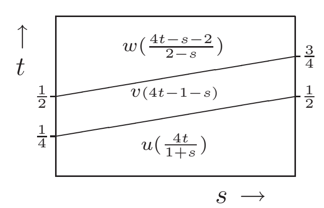
$$u \cdot v(t) = \begin{cases} u(2t) & (0 \leqslant t \leqslant \frac{1}{2}) \\ v(2t-1) & (\frac{1}{2} \leqslant t \leqslant 1) \end{cases}$$

This product is *not* associative: given a third path w from z to a, we have

$$(u \cdot v) \cdot w(t) = \begin{cases} u(4t) & (0 \leqslant t \leqslant \frac{1}{4}) \\ v(4t-1) & (\frac{1}{4} \leqslant t \leqslant \frac{1}{2}) \\ w(2t-1) & (\frac{1}{2} \leqslant t \leqslant 1) \end{cases} \quad \text{but} \quad u \cdot (v \cdot w)(t) = \begin{cases} u(2t) & (0 \leqslant t \leqslant \frac{1}{2}) \\ v(4t-2) & (\frac{1}{2} \leqslant t \leqslant \frac{3}{4}) \\ w(4t-3) & (\frac{3}{4} \leqslant t \leqslant 1) \end{cases}$$

But $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are homotopic.

Define $H(t, s) = \begin{cases} u\left(\frac{4t}{1+s}\right) & \left(0 \leqslant t \leqslant \frac{1+s}{4}\right) \\ v(4t-1-s) & \left(\frac{1+s}{4} \leqslant t \leqslant \frac{2+s}{4}\right) \\ w\left(\frac{4t-s-2}{2-s}\right) & \left(\frac{2+s}{4} \leqslant t \leqslant 1\right) \end{cases}$



Note also that $H(0, s) = x$ and $H(1, s) = a$ for all s.

In general, given two maps between spaces $X \xrightarrow{f,g} Y$ which agree on some $A \subset X$, we say $f \simeq g$ **relative to** A, written $f \simeq g$ rel A, if there is a homotopy $H : X \times I \to Y$ between them with $H(x, t) = f(x)$ for all $x \in A$ and all $0 \leqslant t \leqslant 1$.

So we showed above that $(u \cdot v) \cdot w \simeq u \cdot (v \cdot w)$ rel $\{0, 1\}$.

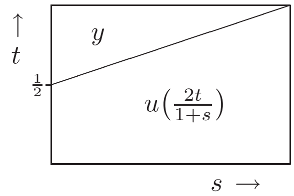
For every $x \in X$, let c_x denote the constant path $c_x(t) = x$ $(0 \leqslant t \leqslant 1)$.

Given a path u, let $\overline{u}$ denote the path $\overline{u}(t) = u(1-t)$.

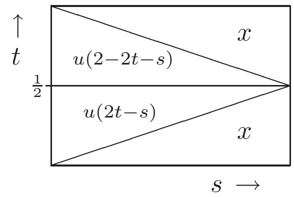
For any u from x to y, we have $u \cdot c_y \simeq u \simeq c_x \cdot u$ rel $\{0, 1\}$, and $u \cdot \overline{u} \simeq c_x$ rel $\{0, 1\}$ and $\overline{u} \cdot u \simeq c_y$ rel $\{0, 1\}$.

To prove these, consider the following homotopies.

$$u \cdot c_y \simeq u, \text{ via } K(t,s) = \begin{cases} u\left(\frac{2t}{1+s}\right) & \left(0 \leqslant t \leqslant \frac{1+s}{2}\right) \\ y & \left(\frac{1+s}{2} \leqslant t \leqslant 1\right) \end{cases}$$



$$u \cdot \overline{u} \simeq c_x, \text{ via } L(t,s) = \begin{cases} x & \left(0 \leqslant t \leqslant \frac{s}{2}\right) \\ u(2t-s) & \left(\frac{s}{2} \leqslant t \leqslant \frac{1}{2}\right) \\ u(2-2t-s) & \left(\frac{1}{2} \leqslant t \leqslant 1-\frac{s}{2}\right) \\ x & \left(1-\frac{s}{2} \leqslant t \leqslant 1\right) \end{cases}$$



1.5 Definition. Let X be a space, and $x \in X$. The **fundamental group** $\Pi_1(X, x)$ is defined to be the set of homotopy classes rel $\{0, 1\}$ of loops in X based at x, with product operation $[u][v] = [u \cdot v]$.

(This is well-defined, since if $u \simeq u'$ rel $\{0, 1\}$ and $v \simeq v'$ rel $\{0, 1\}$, then we can paste the homotopies to get $u \cdot v \simeq u' \cdot v'$ rel $\{0, 1\}$.)

1.6 Theorem.

(i) $\Pi_1(X, x)$ is a group.

(ii) Π_1 is a functor from spaces-with-basepoint to groups.

Proof.

(i) Already done.

(ii) Given based spaces (X, x) and (Y, y), and a continuous map $f : X \to Y$ with $f(x) = y$, we have a mapping $\{\text{loops in } X\} \to \{\text{loops in } Y\}$ given by $u \mapsto f \circ u$.

This respects homotopy rel $\{0, 1\}$, i.e. $u \simeq v$ rel $\{0, 1\} \Rightarrow f \circ u \simeq f \circ v$ rel $\{0, 1\}$, so it induces $f_* : \Pi_1(X, x) \to \Pi_1(Y, y)$, $f_*([u]) = [f \circ u]$.

Also, $f \circ (u \cdot v) = (f \circ u) \cdot (f \circ v)$, so f_* is a group homomorphism.

Finally, given $g : (Y, y) \to (Z, z)$, we have $g \circ (f \circ u) = (g \circ f) \circ u$, so $g_* \circ f_* = (g \circ f)_* : \Pi_1(X, x) \to \Pi_1(Z, z)$, and clearly $(\mathrm{id}_X)_* = \mathrm{id}_{\Pi_1(X,x)}$, and so it is a functor. $\square$

1.7 Remark. Suppose $f, g : (X, x) \to (Y, y)$ are two maps of spaces-with-basepoint. If $f \simeq g$ rel $\{x\}$, i.e. if the homotopy fixes the basepoint, then for any loop u in X based at x, we have $f \circ u \simeq g \circ u$ rel $\{0, 1\}$, so $f_* = g_* : \Pi_1(X, x) \to \Pi_1(Y, y)$.

Note that $\Pi_1(X,x) = \Pi_1(X',x)$, where $X' \subset X$ is the path-component of x (i.e. the set of points reachable by paths from x). If x and y lie in different path-components of X, then there is no relation between $\Pi_1(X,x)$ and $\Pi_1(X,y)$. However:

1.8 Lemma. If x and y are in the same path-component of X, then $\Pi_1(X,x) \cong \Pi_1(X,y)$.

In particular, if X is path-connected, we often write $\Pi_1(X)$ for $\Pi_1(X,x)$.

Proof. Let $v : I \to X$ be a path with $v(0) = x$ and $v(1) = y$. We define $v_\# : \Pi_1(X,x) \to \Pi_1(X,y)$ by $v_\#([u]) = [\overline{v} \cdot u \cdot v]$.

This is well-defined: if $u \simeq u'$ rel $\{0,1\}$ then $(\overline{v} \cdot u) \cdot v \simeq (\overline{v} \cdot u') \cdot v$ rel $\{0,1\}$.

$v_\#$ is a homomorphism: given another loop w, we have

$$v_\#([u])v_\#([w]) = [\overline{v} \cdot u \cdot v \cdot \overline{v} \cdot w \cdot v] = [\overline{v} \cdot u \cdot c_x \cdot w \cdot v] = [\overline{v} \cdot u \cdot w \cdot v] = v_\#([u][w])$$

And $v_\#$ has an inverse, namely $\overline{v}_\#$:

$$\overline{v}_\# v_\#([u]) = [v \cdot \overline{v} \cdot u \cdot v \cdot \overline{v}] = [c_x \cdot u \cdot c_x] = [u]$$

$\square$

1.9 Remark. The isomorphism $v_\#$ may depend on the choice of the path v. If w is another path from x to y, then

$$\overline{w}_\# v_\#([u]) = [w \cdot \overline{v} \cdot u \cdot v \cdot \overline{w}] = [v \cdot \overline{w}]^{-1}[u][v \cdot \overline{w}]$$

so $\overline{w}_\# v_\#$ is conjugation by the element $[v \cdot \overline{w}]$ of $\Pi_1(X,x)$.

Hence if $\Pi_1(X,x)$ is abelian then we have $\overline{w}_\# v_\# = \text{id}$ and hence $v_\# = w_\#$, but in general it need not be.

(Note also that if $v \simeq w$ rel $\{0,1\}$ then $\overline{w} \cdot v \simeq c_x$ rel $\{0,1\}$, so $v_\# = w_\#$.)

1.10 Corollary. If $f : X \to Y$ is (part of) a homotopy equivalence, then $f_* : \Pi_1(X,x) \to \Pi_1(Y,f(x))$ is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse for f. By restricting a homotopy $\text{id}_X \simeq gf$ to the point x, we obtain a path v from x to $gf(x)$ in X.

Also, for any loop u based at x, we have a homotopy $H : I \times I \to X$ from u to gfu with $H(0,s) = H(1,s) = v(s)$ for all $0 \leqslant s \leqslant 1$.

By pasting suitable homotopies on to this one, we get a homotopy rel $\{0,1\}$ from u to $v \cdot gfu \cdot \overline{v}$. So the composite

$$\Pi_1(X,x) \xrightarrow{f_*} \Pi_1(Y,f(x)) \xrightarrow{g_*} \Pi_1(X,gf(x)) \xrightarrow{\overline{v}_\#} \Pi_1(X,x)$$

is the identity. In particular, this shows that f_* is injective. By symmetry, g_* is also injective. And $v_\#$ is an isomorphism by **1.8**.

Hence f_* is surjective and therefore an isomorphism. $\square$

In particular, if X is contractible then $\Pi_1(X) \cong \Pi_1(\{x\})$ is the trivial group.

If X is path-connected and $\Pi_1(X)$ is trivial, we say that X is **simply connected**.

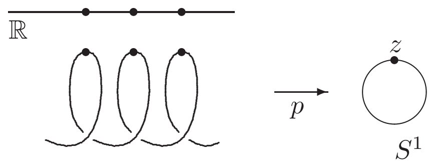
Chapter 2 : Covering Spaces

2.1 Definition. By a **trivial covering projection** we mean a projection map $X \times D \to X$, where D is a discrete space.

A **covering projection** $p : Y \to X$ is a continuous map which is 'locally a trivial covering projection'. That is, such that there is an open covering $\{U_\alpha : \alpha \in A\}$ of X, and for each α a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times D_\alpha$ (D_α discrete) such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "h_\alpha"] \arrow[dr, "p|_{p^{-1}(U_\alpha)}"'] & & U_\alpha \times D_\alpha \arrow[dl, "p_1"] \\ & U_\alpha & \end{tikzcd}$$

Example. Consider the mapping $p : \mathbb{R} \to S^1 \subset \mathbb{C}$, $t \mapsto e^{2\pi i t}$. For every $z \in S^1$, if we take a small open neighbourhood U of z, we see that $p^{-1}(U) \cong U \times \mathbb{Z}$ (with the discrete topology on $\mathbb{Z}$).



We say a subset Y of X is **evenly covered** by p if there is a homeomorphism $p^{-1}(Y) \to Y \times D$, with D discrete, making
$$\begin{tikzcd} p^{-1}(Y) \arrow[rr] \arrow[dr] & & Y \times D \arrow[dl] \\ & Y & \end{tikzcd}$$
commute.

2.2 Lebesgue covering theorem. Let X be a compact metric space ('think closed and bounded subset of $\mathbb{R}^n$'), and $\{U_\alpha : \alpha \in A\}$ an open cover of X.

Then there exists $\varepsilon > 0$ such that for all $x \in X$ there is $\alpha \in A$ with $B(x, \varepsilon) \subset U_\alpha$.

Such an ε is called a **Lebesgue number** for the covering.

Proof. For each $x \in X$, choose $\varepsilon_x > 0$ such that $B(x, 2\varepsilon_x)$ is contained in some U_α. The balls $\{B(x, \varepsilon_x) : x \in X\}$ form an open covering of X, so by compactness there is a finite subcover, $\{B(x_1, \varepsilon_{x_1}), \ldots, B(x_n, \varepsilon_{x_n})\}$, say.

Let $\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_n}\}$. Then for any $y \in X$ there exists i such that $y \in B(x_i, \varepsilon_{x_i})$, and so $B(y, \varepsilon) \subset B(y, \varepsilon_{x_i}) \subset B(x_i, 2\varepsilon_{x_i}) \subset U_\alpha$. $\square$

2.3 Theorem. Let $p : Y \to X$ be a covering projection.

(i) **Path lifting property.** Suppose we have a path $u : I \to X$ and a point $y \in Y$ such that $p(y) = u(0)$. Then there exists a unique path $\widetilde{u} : I \to Y$ such that $p \circ \widetilde{u} = u$ and $\widetilde{u}(0) = y$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ I \arrow[ur, "\widetilde{u}"] \arrow[r, "u"'] & X \end{tikzcd}$$

(ii) **Homotopy lifting property.** Let K be a compact metric space, and let $f, g : K \to X$ be two continuous maps, $H : K \times I \to X$ a homotopy from f to g, and $\widetilde{f} : K \to Y$ a continuous map such that $p\widetilde{f} = f$.

Then there exists a unique homotopy $\widetilde{H} : K \times I \to Y$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(k, 0) = \widetilde{f}(k)$ for all $k \in K$

Note that (i) is the special case $K = \{x\}$ of (ii).

Proof.

(i) Let $\mathcal{U}$ be an open covering of X by sets U which are evenly covered by p. Consider the open covering $\{u^{-1}(U) : U \in \mathcal{U}\}$ of I.

Since I is a compact metric space, we may choose n such that $\frac{1}{n}$ is a Lebesgue number for the covering, and subdivide I into the closed intervals $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ for $1 \leqslant i \leqslant n$. Let $u_i = u|_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}$, and let $X_i = \left\{u_i(t) : \frac{i-1}{n} \leqslant t \leqslant \frac{i}{n}\right\}$. Then X_i is evenly covered by p.

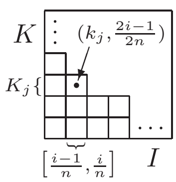
Let $h_1 : p^{-1}(X_1) \to X_1 \times D_1$ be a homeomorphism, where D_1 is discrete. Write $h_1(y) = (u(0), d)$ for some $d \in D_1$. Note that if $\widetilde{u}_1$ is any lifting of u_1 starting at y, then the composition

$$[0, \tfrac{1}{n}] \xrightarrow{\widetilde{u}_1} p^{-1}(X_1) \xrightarrow{h} X_1 \times D_1 \xrightarrow{p_2} D_1$$

(where p_2 is projection onto the second component) must be constant, since $[0, \frac{1}{n}]$ is connected. Hence if we define $\widetilde{u}_1(t) = h^{-1}(u(t), d)$ for $0 \leqslant t \leqslant \frac{1}{n}$, then $\widetilde{u}_1$ is the unique continuous lifting of u_1 starting at y.

Similarly, construct a unique continuous lifting $\widetilde{u}_2$ of u_2 with $\widetilde{u}_2(\frac{1}{n}) = \widetilde{u}_1(\frac{1}{n})$, and so on. Then we may patch these together to obtain a continuous map $\widetilde{u} : I \to Y$ such that $\widetilde{u}|_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} = \widetilde{u}_i$, which is the unique continuous of u starting at y.

(ii) Let $\mathcal{U}$ be as before. Consider the open covering $\{H^{-1}(U) : U \in \mathcal{U}\}$ of $K \times I$ (which is compact), and choose n such that $\frac{1}{n}$ is a Lebesgue number for this covering.



Pick a finite cover of K by open balls $B(k_1, \frac{1}{2n}), \ldots, B(k_m, \frac{1}{2n})$, and let $K_1, \ldots, K_m$ be the corresponding closed balls.

For each (i, j), the set $K_j \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$ is contained ('using Pythagorean distance') in the ball $B\big((k_j, \frac{2i-1}{2n}), \frac{1}{n}\big)$, and so if we set

$$H_{i,j} = H|_{K_j \times \left[\frac{i-1}{n}, \frac{i}{n}\right]} \quad \text{and} \quad X_{i,j} = \left\{H_{i,j}(k,t) : (k,t) \in K_j \times \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right]\right\}$$

then $X_{i,j}$ is evenly covered by p.

Set $i = 1$ and consider a particular j. Let $h_{1,j} : p^{-1}(X_{1,j}) \to X_{1,j} \times D_{1,j}$ be a homeomorphism, and set $h_{1,j}\big(\widetilde{f}(k)\big) = \big(f(k), d(k)\big)$.

Then if we put $\widetilde{H}_{1,j}(k,t) = h_{1,j}^{-1}\big(H_{1,j}(k,t), d(k)\big)$, it is the unique continuous lifting of $H_{1,j}$ to a map $K_j \times [0, \frac{1}{n}] \to Y$ such that $\widetilde{H}_{1,j}(k, 0) = \widetilde{f}(k)$ for $k \in K_j$.

Note that if $K_j \cap K_{j'} \neq \emptyset$, the restrictions of $\widetilde{H}_{1,j}$ and $\widetilde{H}_{1,j'}$ to $(K_j \cap K_{j'}) \times [0, \frac{1}{n}]$ must agree, by the uniqueness of the lifting of $H|_{(K_j \cap K_{j'}) \times [0, \frac{1}{n}]}$.

So we can patch the $\widetilde{H}_{1,j}$ together to get a unique $\widetilde{H}_1 : K \times [0, \frac{1}{n}] \to Y$ with $\widetilde{H}_1(k, 0) = \widetilde{f}(k)$ for all k and $p \circ \widetilde{H}_1(k,t) = H(k,t)$ for all k and all $0 \leqslant t \leqslant \frac{1}{n}$.

Now, for $2 \leqslant i \leqslant n$, construct a lifting $\widetilde{H}_i$ on $K \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$ and patch these together as before. $\square$

2.4 Corollary.

(i) If $p : Y \to X$ is a covering projection, then for any $y \in Y$ the homomorphism $p_* : \Pi_1(Y, y) \to \Pi_1(X, p(y))$ is injective.

(ii) If $p : Y \to X$ is a surjective covering projection and Y is simply connected, then for any $x \in X$ there is a bijection $p^{-1}(x) \to \Pi_1(X, x)$. (This gives us the number of elements in the group.)

Proof.

(i) We have to show that if u and v are loops in Y based at y and $pu \simeq pv$ rel $\{0, 1\}$, then $u \simeq v$ rel $\{0, 1\}$.

Let $H : I \times I \to X$ be a homotopy between pu and pv. Lift H to a homotopy $\widetilde{H}$ with $\widetilde{H}(0, s) = y$ for all s. Then necessarily $\widetilde{H}(t, 0) = u(t)$ and $\widetilde{H}(t, 1) = v(t)$ for all t, since they are liftings of pu and pv starting at y. Similarly, we must have $\widetilde{H}(1, s) = y$ for all s.

So $\widetilde{H}$ is a homotopy from u to v rel $\{0, 1\}$, as required.

(ii) Pick y_0 in $p^{-1}(x)$. Any loop in X based at x can be lifted to a path in Y starting at y_0, and ending at some point of $p^{-1}(x)$. Moreover, if loops u and v satisfy $u \simeq v$ rel $\{0, 1\}$ then we can lift a homotopy between them to a homotopy between $\widetilde{u}$ and $\widetilde{v}$ rel $\{0, 1\}$.

Hence in particular $\widetilde{u}(1) = \widetilde{v}(1)$. So the mapping $[u] \mapsto \widetilde{u}(1)$ is a well-defined mapping $\Pi_1(X, x) \to p^{-1}(x)$.

Since Y is (simply connected and hence) path-connected, for any $y \in p^{-1}(x)$ there is a path $\widetilde{u}$ from y_0 to y, and $p\widetilde{u} = u$ is a loop in X based at x.

If $\widetilde{u}$, $\widetilde{v}$ are two such paths, then $\widetilde{u} \cdot \overline{\widetilde{v}} \simeq c_{y_0}$ rel $\{0, 1\}$ and $\overline{\widetilde{v}} \cdot \widetilde{u} \simeq c_y$ rel $\{0, 1\}$, so

$$\widetilde{v} \;\simeq\; \widetilde{v} \cdot c_y \;\simeq\; \widetilde{v} \cdot \overline{\widetilde{v}} \cdot \widetilde{u} \;\simeq\; c_{y_0} \cdot \widetilde{u} \;\simeq\; \widetilde{u} \text{ rel } \{0, 1\}$$

So $p\widetilde{u} \simeq p\widetilde{v}$ rel $\{0, 1\}$. Hence we have a well-defined mapping $p^{-1}(x) \to \Pi_1(X, x)$ which is inverse to the one defined above. □

Consider the covering projection $\mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. If we take 1 as base point for S^1, then $p^{-1}(1) = \mathbb{Z}$. In this case it's easy to show that the group operation on $\Pi_1(S^1, 1)$ corresponds to addition in $\mathbb{Z}$.

2.5 Definition. Let $p : Y \to X$ be a covering projection. By a **covering translation** of Y, we mean a homeomorphism $\theta : Y \to Y$ such that $p\theta = p$. Covering translations are also called **deck transformations**.

In the example of S^1, the covering translations are the maps $t \mapsto t + n$, $n \in \mathbb{Z}$, and they form a group isomorphic to $(\mathbb{Z}, +)$.

We say a space X is **locally path-connected** if given any $x \in X$ and any open neighbourhood U of x, we can find a smaller open neighbourhood V of x such that any two points in V may be joined by a path taking values in U.

2.6 Proposition. Suppose we have a covering projection $p : Y \to X$ where Y is simply connected and X is locally path-connected. Then given any two points $y_1, y_2 \in Y$ with $p(y_1) = p(y_2)$ there is a unique covering translation $\theta : Y \to Y$ with $\theta(y_1) = y_2$.

Proof. Given any $z \in Y$, choose a path u in Y with $u(0) = y_1$, $u(1) = z$. Then let u^* be the unique lifting of pu to a path in Y starting at y_2 and define $\theta(z) = u^*(1)$.

This is well-defined. If v is another path from y_1 to z in Y, then $u \simeq v$ rel $\{0, 1\}$ since Y is simply connected, so $pu \simeq pv$ rel $\{0, 1\}$. So $u^* \simeq v^*$ rel $\{0, 1\}$, so in particular $u^*(1) = v^*(1)$.

$\theta(y_1) = y_2$, since if $z = y_1$ then we can take u to be c_{y_1} and then $u^* = c_{y_2}$.

It is clear that $p\theta = p$.

θ is bijective, as it has an inverse defined in the same way, by interchanging y_1 and y_2.

To show θ is continuous, it is enough to show that $\theta|_{p^{-1}(U)}$ is continuous for an evenly covered open set $U \subset X$. For simplicity, identify $p^{-1}(U)$ with $U \times D$, where D is discrete.

Pick a point $z = (x, d)$ in $U \times D$. Then $\theta(z) = (x, d')$. Take $U \times \{d'\}$ as an open neighbourhood of $\theta(z)$ in Y. Choose an open V in X with $x \in V \subset U$ satisfying the definition of local path-connectedness.

For each $z' \in V \times \{d\}$ we can choose a path v from z to z' in $U \times \{d\}$, and then choose $u \cdot v$ as a path from y_1 to z', where u was our chosen path from y_1 to z.

Then clearly pv lifts to a path starting at $\theta(z)$ which lies within $U \times \{d'\}$, and so $\theta(z') \in V \times \{d'\}$. Hence $V \times \{d\} \subset \theta^{-1}(V \times \{d'\})$ – in fact, we have equality here.

So θ is continuous, and similarly θ^{-1} is continuous.

Uniqueness is obvious: if θ is a covering translation with $\theta(y_1) = y_2$, then for any path u starting at y_1, θu is a path starting at y_2 and lying over the same path in X. □

2.7 Corollary. Let X be locally path-connected and suppose we have a surjective covering projection $p : Y \to X$ such that Y is simply connected. (We call such Y a **universal cover** of X.)

Then for any $x \in X$, $\Pi_1(X, x)$ is isomorphic to the group G of covering translations $\theta : Y \to Y$.

Proof. Pick $y_0 \in p^{-1}(x)$. By **2.4**(ii) we have a bijection $\Pi_1(X, x) \to p^{-1}(x)$, and by **2.6** we have a bijection $p^{-1}(x) \to G$ sending $y_1 \in p^{-1}(x)$ to the unique θ with $\theta(y_0) = y_1$.

Let u, v be loops in X based at x. Then let $\widetilde{u}$ be the unique lifting of u starting at y_0, let $y_1 = \widetilde{u}(1)$, let $\widetilde{v}$ be the unique lifting of v starting at y_1, and let $y_2 = \widetilde{v}(1)$.

If θ is the covering translation sending y_0 to y_1 then $\theta^{-1}(\widetilde{v})$ starts at y_0, so $\theta^{-1}(y_2)$ corresponds to $[v]$ under the first bijection.

Let φ be the covering translation sending y_0 to $\theta^{-1}(y_2)$. Then $\theta \circ \varphi$ sends y_0 to y_2, so it corresponds to $[u \cdot v]$ under the composite bijection. □

2.8 Examples.

(a) S^1 has a universal cover $\mathbb{R} \to S^1$, so $\Pi_1(S^1) \cong \mathbb{Z}$.

(b) Consider the torus T, which we can think of as the quotient of the unit square $I \times I$ by the equivalence relation which identifies $(t, 0)$ with $(t, 1)$ for all $0 \leqslant t \leqslant 1$, and $(0, s)$ with $(1, s)$ for all $0 \leqslant s \leqslant 1$.

There is a covering projection $\mathbb{R}^2 \to T$ which sends $\mathbb{R}^2 \xrightarrow{p} T$, by sending (x, y) to $(x \bmod 1, y \bmod 1)$. We have $p^{-1}\big((0,0)\big) = \mathbb{Z} \times \mathbb{Z}$, and the covering translations are maps of the form $(x, y) \mapsto (x + m, y + n)$ with $m, n \in \mathbb{Z}$. So $\Pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

(c) Real projective space, $\mathbb{RP}^n$, is the quotient of $\mathbb{R}^{n+1} \backslash \{0\}$ by the equivalence relation which identifies x and y if $x = ty$ for some scalar t. Equivalently, it is the quotient of S^n by the equivalence relation which identifies x and y if $x = \pm y$.

The quotient map $S^n \xrightarrow{q} \mathbb{RP}^n$ is a covering projection: every point of $\mathbb{RP}^n$ has a neighbourhood U homeomorphic to an open ball in $\mathbb{R}^n$, whose inverse image under q is homeomorphic to $U \times \{\pm 1\}$.

In chapter 3, we will see that S^n is simply connected for $n \geqslant 2$, so it's a universal cover of $\mathbb{RP}^n$, and so $\Pi_1(\mathbb{RP}^n)$ is cyclic of order 2. The covering translations are id_{S^n} and the **antipodal map** $x \mapsto -x$.

For $n = 1$, $\mathbb{RP}^1$ is homeomorphic to S^1, so $\Pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$, but $q_* : \Pi_1(S^1) \to \Pi_1(\mathbb{RP}^1)$ is multiplication by 2, so it's not an isomorphism.

(d) Consider the space $S^1 \vee S^1 \ldots$

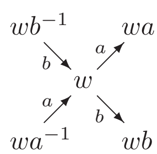
Given a set A of generators, we define the **free group** FA on A to be the set of equivalence classes of formal (finite) products $x_1 \ldots x_n$, where for each i, x_i is either a member a_i of$A$, or a symbol a_i^{-1} where $a_i \in A$.

We identify two formal products if we can get from one to the other by deleting or inserting adjacent pairs of the form aa^{-1} or $a^{-1}a$. Equivalently, we could consider FA as the set of **reduced words**, i.e. those containing no instances of aa^{-1} or $a^{-1}a$.

The product of two words v, w is obtained by concatenating them (and cancelling as necessary). The identity element is the empty word ( ), and the inverse w^{-1} of w is obtained by inverting all of its letters and reversing their order.

Given any function $f : A \to G$, where G is a group, we can uniquely extend f to a homomorphism $\widetilde{f} : FA \longrightarrow G$ by setting $\widetilde{f}(x_1 \ldots x_n) = f(x_1) \ldots f(x_n)$, as computed in G.

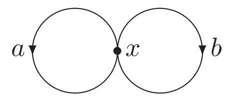
Now let $F_2 = F\{a, b\}$. The **Cayley graph** C of F_2 has the elements of F_2 as vertices, and for each $w \in F_2$ we have (directed) edges labelled a from w to wa, and labelled b from w to wb. Thus each vertex has two incoming and two outgoing eges.



We make C into a topological space, namely the quotient of $F_{\mathrm{disc}} \cup \big((F \times \{a, b\})_{\mathrm{disc}} \times I\big)$ by the equivalence relation which identifies $(w, a, 0)$ and $(w, b, 0)$ with w, identifies $(w, a, 1)$ with wa, and identifies $(w, b, 1)$ with wb.

2.8 continued.

(d) Let $p : C \to S^1 \vee S^1$ be the map sending all vertices to x and wrapping all edges labelled a around the left-hand circle, and all edges labelled b around the right-hand circle. Then p is easily seen to be a covering projection.



C is simply connected (in fact contractible). For each vertex w there is a unique path from ( ) to w that doesn't 'backtrack'. E.g., for $w = aab^{-1}a^{-1}b$, we have

$$(\ ) \xrightarrow{a} a \xrightarrow{a} aa \xleftarrow{b} aab^{-1} \xleftarrow{a} aab^{-1}a^{-1} \xrightarrow{b} aab^{-1}a^{-1}b = w$$

If we define $H : C \times I \to C$ by saying that $t \mapsto H(x,t)$ moves x at uniform speed along this path towards ( ), then H is a homotopy from id_C to the constant map with value ( ).

The covering translations are of the form $w \mapsto vw$, $(w, {}^a_b, t) \mapsto (vw, {}^a_b, t)$ for some $v \in F_2$, and they form a group isomorphic to F_2. So $\Pi_1(S^1 \vee S^1) \cong F_2$.

2.9 Remark. From the knowledge that $\Pi_1(S^1) \cong \mathbb{Z}$, we can give a proof of the Fundamental Theorem of Algebra: that every non-constant polynomial over $\mathbb{C}$ has a root.

Let $f(z) = z^n + g(z)$ be a polynomial, where g is of degree $\leqslant n-1$, and suppose that f has no roots. Then $h(z) = \frac{f(z)}{|f(z)|}$ defines a continuous map $\mathbb{C} \cong \mathbb{R}^2 \to S^1$.

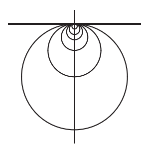
Choose R such that $|g(z)| < R^n$ whenever $|z| = R$, and then $z \mapsto f(z)$ and $z \mapsto z^n$ are homotopic as maps $\{z : |z| = R\} \to \mathbb{C} \setminus \{0\}$ by a linear homotopy.

Hence also $h(z)$ and $z \mapsto \frac{z^n}{R^n}$ are homotopic as maps into S^1. But $z \mapsto \frac{z^n}{R^n}$ induces the mapping $a \mapsto na$ on $\Pi_1(S^1)$, so h also induces this mapping.

But h factors as $\{z : |z| = R\} \hookrightarrow \mathbb{C} \to S^1$, and $\Pi_1(\mathbb{C}) = \{0\}$, so h_* is the zero map. $\times\!\!\!\!\times$

Does every path-connected and locally path-connected space have a universal cover? No.

2.10 Example. The **Hawaiian earring** is the subspace $H = \bigcup_{n=1}^{\infty} C_n$ of $\mathbb{R}^2$, where C_n is the circle with centre $(0, -\frac{1}{n})$ and radius $\frac{1}{n}$.



Suppose $p : X \to H$ is a surjective covering projection. Let U be an evenly covered neighbourhood of $(0,0)$. Then for sufficiently large n, U contains the whole of C_n.

If $x = \big((0,0), d\big)$ is a point of $p^{-1}(U) = U \times D$ lying over $(0,0)$, then $U \times \{d\}$ is a neighbourhood of x.

So there is a loop u in $U \times \{d\} \subset X$ going once around some $C_n \times \{d\}$. Then $[pu]$ is a non-identity element of $\Pi_1\big(H, (0,0)\big)$, so $[u]$ is a non-identity element of $\Pi_1(X, x)$. So X is not simply connected.

2.11 Theorem.

(i) Suppose that X is path-connected and that X has an open covering by simply connected subspaces. Then there exists a covering projection $p : Y \to X$ where Y is simply connected.

(ii) If X is also locally path-connected then such a Y is unique up to homeomorphism.

Proof. (Non-examinable.)

(i) Let $\{U_\alpha : \alpha \in A\}$ be an open covering of X by simply connected subspaces. Pick $x_\alpha \in U_\alpha$ for each α. Also pick a particular $\alpha_0 \in A$, and write x_0 for x_{α_0}. Let D_α be the set of homotopy classes rel $\{0, 1\}$ of paths from x_0 to x_α in X, equipped with the discrete topology.

We'll set $p^{-1}(U_\alpha) = U_\alpha \times D_\alpha$ for each α.

Take Y to be $\bigcup_{\alpha \in A}(U_\alpha \times D_\alpha)/\sim$, where the equivalence relation $\sim$ identifies $(y, [u_\alpha]) \in U_\alpha \times D_\alpha$ with $(y, [u_\beta]) \in U_\beta \times D_\beta$ (for any $y \in U_\alpha \cap U_\beta$) iff $u_\alpha \cdot v_\alpha \simeq u_\beta \cdot v_\beta$ rel $\{0, 1\}$, where v_α is a path from x_α to y lying in U_α, and v_β is a path from x_β to y lying in U_β.

(Note that this is well-defined, since U_α and U_β are simply connected.)

The projection $p : Y \to X$ is induced by the first projections $U_\alpha \times D_\alpha \to U_\alpha \hookrightarrow X$. It's clear that $p^{-1}(U_\alpha) \cong U_\alpha \times D_\alpha$, so p is a covering projection. Note that $D_\alpha \neq \emptyset$ for each α (since X is path-connected) and hence $p : Y \to X$ is surjective.

Now let u be a path in X starting at x_0, and let $\widetilde{u}$ be the unique lifting of u to a path in Y starting at $(x_0, [c_{x_0}])$. If $u(1) = y \in U_\alpha$ then we claim $\widetilde{u}(1) = (y, [u \cdot \overline{v}])$, where v is any path in U_α from x_α to y.

To prove the claim, use the Lebesgue covering theorem to subdivide $I = [0, 1]$ into intervals $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, $1 \leqslant i \leqslant n$, each of which is mapped into a single U_α, and show by induction that $\widetilde{u}(\frac{i}{n}) = \big(u(\frac{i}{n}), [u_i \cdot \overline{v}_i]\big)$, where $u_i(t) = u(\frac{i}{n}t)$ and v_i is a path from x_{α_i} to $u(\frac{i}{n})$ in U_{α_i}, for suitable α_i.

In particular, if u is a loop in X based at x_0, then $\widetilde{u}$ will be a loop in Y iff $u \simeq c_{x_0}$ rel $\{0, 1\}$. So we can lift the homotopy to obtain $\widetilde{u} \simeq c_{\widetilde{x}_0}$ rel $\{0, 1\}$.

Hence $\Pi_1(Y, \widetilde{x}_0)$ is trivial.

(ii) Now suppose we are given two covering projections $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ with Y_1 and Y_2 simply connected.

Pick $y_1 \in Y_1$ and $y_2 \in Y_2$ with $p_1(y_1) = p_2(y_2)$. Then by the method of **2.6** we can construct a homeomorphism $\theta : Y_1 \to Y_2$ with $p_2\theta = p_1$ and $\theta(y_1) = y_2$. □

2.12 Corollary. If X is locally path-connected and simply connected then any covering projection $p : Y \to X$ is trivial (i.e. $Y \cong X \times D$ with D discrete).

Proof. Y inherits local path-connectedness from X, so its path-components are open. Let Y_α be a path-component of Y. Then $p|_{Y_\alpha}$ is still a covering projection, so by **2.4**(i), $\Pi_1(Y_\alpha) \to \Pi_1(X)$ is injective and hence $\Pi_1(Y_\alpha)$ is trivial.

Hence $p|_{Y_\alpha}$ is a universal covering of X. But so is id_X and so $p|_{Y_\alpha}$ is a homeomorphism.

Now let D be the set of path-components of Y, with the discrete topology. Then we have a homeomorphism $h : Y \to X \times D$ given by $h(y) = \big(p(y), [y]\big)$, where $[y]$ denotes the path-component containing y. □

2.13 Theorem. Suppose X is path-connected and locally path-connected, and has a covering by simply connected open sets. Then homeomorphism classes over X of covering projections $p : Y \to X$ with Y path-connected correspond bijectively with conjugacy classes of subgroups of $\Pi_1(X)$.

Proof. Pick $x \in X$. Given a covering projection $p : Y \to X$, for any $y \in p^{-1}(x)$ we have an injective homomorphism $p_* : \Pi_1(Y, y) \to \Pi_1(X, x)$ by **2.4**(i), and hence have a subgroup of $\Pi_1(X, x)$.

If y' is another point of $p^{-1}(x)$, let v be a path in Y from y to y'. Then the isomorphism $v_\# : \Pi_1(Y, y) \to \Pi_1(Y, y')$ of **1.8** is mapped by p_* to the operation of conjugation by $[pv]$ in $\Pi_1(X, x)$. So $p_*\Pi_1(Y, y)$ and $p_*\Pi_1(Y, y')$ are conjugate subgroups of $\Pi_1(X, x)$.

Moreover every conjugate of $p_*\Pi_1(Y, y)$ occurs in this way: if u is any loop in X based at x then we can lift it to a path in Y, say $\widetilde{u}$, starting at y, and then $[u]^{-1}\big(p_*\Pi_1(Y, y)\big)[u]$ is $p_*\Pi_1(Y, \widetilde{u}(1))$.

Conversely, let $p : \widetilde{X} \to X$ be the universal covering of X as constructed in **2.11**. Choose $\widetilde{x} \in p^{-1}(x)$, then by **2.7** we have an isomorphism from $\Pi_1(X, x)$ to the group G of covering translations $\widetilde{X} \to \widetilde{X}$.

Let H be any subgroup of G. Then H acts on $\widetilde{X}$ by homeomorphisms and for any $y \in \widetilde{X}$ we can find a neighbourhood V of y such that the sets $h[V]$, $h \in H$, are pairwise disjoint. (Specifically, take V to be a neighbourhood of the form $U \times \{d\}$ where U is an evenly covered neighbourhood of $p(y)$ and we identify $p^{-1}(U)$ with $U \times D$ for D discrete.)

Now let $Y = \widetilde{X}/H$ be the set of orbits of the action of H, topologised as a quotient space of $\widetilde{X}$. Then $p : \widetilde{X} \to X$ factors as $\widetilde{X} \xrightarrow{q} Y \xrightarrow{r} X$, and both q and r are covering projections.

(Given $y \in \widetilde{X}$, choose V as above and then $q[V]$ is open in Y since $q^{-1}(q[V]) = \bigcup_{h \in H} h[V]$ is open and it's evenly covered by q. And any open set U in X which is evenly covered by p is also evenly covered by r.)

Moreover, the covering translations of $\widetilde{X}$ relative to q are exactly the elements of H, so $\Pi_1\big(Y, q(\widetilde{x})\big) \cong H$, and $r_* : \Pi_1\big(Y, q(\widetilde{x})\big) \to \Pi_1(X, x)$ maps $\Pi_1\big(Y, q(\widetilde{x})\big)$ to the right subgroup of $\Pi_1(X, x)$.

It remains to show that every covering projection $r : Y \to X$ with Y connected occurs as a quotient of $\widetilde{X}$. There are two ways of doing this.

(a) Observe that Y is locally path-connected and has an open covering by simply connected subsets, so it has a universal cover $q : \widetilde{Y} \to Y$. Then the composite $rq : \widetilde{Y} \to Y \to X$ is a covering projection and hence must be homeomorphic to $p : \widetilde{X} \to X$. Thus Y is homeomorphic to a quotient space of $\widetilde{X}$

(b) Alternatively, we can construct $q : \widetilde{X} \to Y$ by the method of **2.6**. That is, choose $\widetilde{x} \in p^{-1}(x)$ and $y \in r^{-1}(x)$, and then define q by choosing a path u from $\widetilde{x}$ to z in $\widetilde{X}$, lifting pu to a path in Y starting at y and taking the end-point of this path. Then q is still surjective (though no longer bijective) since Y is path-connected, and it's still continuous and a quotient map. $\square$

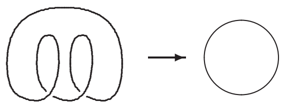
Example. Let $X = S^1$.

Then $\Pi_1(X) \cong \mathbb{Z}$ and its non-trivial subgroups are of the form $n\mathbb{Z}$, $n \geqslant 1$.

Corresponding to the trivial subgroup we have the universal cover $\mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$.

Corresponding to the subgroup $n\mathbb{Z}$ we have the n-fold covering $S^1 \to S^1$, $z \mapsto z^n$.

E.g. $n = 3$.



In particular, every connected covering space of S^1 is homeomorphic to either $\mathbb{R}$ or S^1.

Chapter 3 : The Seifert–Van Kampen Theorem

We've seen that for any set A there is a free group FA generated by A, with the property that for any group G, homomorphisms $FA \to G$ correspond bijectively to arbitrary functions $A \to G$. By a **relation** on A we mean a formal equation $(w_1 = w_2)$, where w_1, w_2 are elements of FA.

3.1 Definition. Let R be a set of relations on A. By the group **generated by A subject to R** we mean the quotient of FA by the smallest normal subgroup N containing $w_1 w_2^{-1}$ for all the equations $(w_1 = w_2)$ in R, i.e. by the subgroup generated by all conjugates $x^{-1} w_1 w_2^{-1} x$ with $x \in FA$ and $(w_1 = w_2) \in R$.

We write $\langle A \,|\, R \rangle$ for this group.

3.2 Lemma. For any group G, homomorphisms $\langle A \,|\, R \rangle \to G$ correspond bijectively to functions $f : A \to G$ satisfying $\widetilde{f}(w_1) = \widetilde{f}(w_2)$ for all $(w_1 = w_2) \in R$, where $\widetilde{f}$ is the unique extension of f to a homomorphism $FA \to G$.

Proof. By definition, homomorphisms $\widetilde{f} : \langle A \,|\, R \rangle \to G$ correspond to homomorphisms $\widetilde{f} : FA \to G$ whose kernel contains N; equivalently, such that $\widetilde{f}(w_1 w_2^{-1}) = e_G$ for all $(w_1 = w_2) \in R$; equivalently, such that $\widetilde{f}(w_1) = \widetilde{f}(w_2)$. $\square$

Note that this property determines $\langle A \,|\, R \rangle$ up to isomorphism. If we have two groups F_1, F_2 satisfying this condition, then $\mathrm{id} : F_1 \to F_1$ corresponds to a mapping $A \to F_1$ satisfying the relations, and hence to a homomorphism $F_2 \to F_1$. Similarly, we get a homomorphism $F_1 \to F_2$, and these are inverse to each other.

We say a given group G has the **presentation** $\langle A \,|\, R \rangle$ if there is a mapping $A \to G$ satisfying the relations in R, such that the induced homomorphism $\langle A \,|\, R \rangle \to G$ is an isomorphism.

Every group has a presentation: given G, take G itself as a set of generators, and take $R = \{(gh = k) : gh = k \in G\}$. Then clearly given homomorphisms $\langle G \,|\, R \rangle \to H$ correspond to (are) homomorphisms $G \to H$.

We say that G is **finitely presented** if it has a presentation $\langle A \,|\, R \rangle$ with A and R finite. It's not always obvious whether a group presentation $\langle A \,|\, R \rangle$ defines a non-trivial group, nor when two different presentations define isomorphic groups. However, we can obtain information about a presentation $\langle A \,|\, R \rangle$ by finding particular groups with generators labelled by A and satisfying R.

Example. Consider the presentation $\langle a, b \,|\, b^{-1}ab = a^{-1} \rangle$.

The dihedral group D_{2n} is generated by elements a (of order n) and b (of order 2) satisfying the relation $b^{-1}ab = a^{-1}$, and so there is a surjective homomorphism $\langle a, b \,|\, b^{-1}ab = a^{-1} \rangle \to D_{2n}$ for all n.

From this we can deduce that $\langle a, b \,|\, b^{-1}ab = a^{-1} \rangle$ is infinite and non-abelian.

3.3 Definition.

(a) Let G and H be two groups. The **free product** $G * H$ is characterised by the fact that for any K, homomorphisms $G * H \to K$ correspond to pairs of homomorphisms $(G \to K, H \to K)$.

To construct it, let $\langle A \,|\, R\rangle$ and $\langle B \,|\, S\rangle$ be presentations of G and H respectively, where $A \cap B = \emptyset$. Then $\langle A \cup B \,|\, R \cup S\rangle$ has the required property.

Note that if $G \cong FA$ and $H \cong FB$ then $G * H \cong F(A \cup B)$.

(b) Let $f : K \to G$, $g : K \to H$ be two group homomorphisms with common domain K. The **free product of** G **and** H **with amalgamation of** K, written $G *_K H$, is the quotient of $G * H$ by the smallest subgroup containing $f(k)g(k)^{-1}$ for all $k \in K$.

Equivalently, if G and H have presentations $\langle A \,|\, R\rangle$ and $\langle B \,|\, S\rangle$ as before, then $G *_K H$ has presentation $\langle A \cup B \,|\, R \cup S \cup \{(f(k) = g(k)) : k \in K\}\rangle$.

(Note: we could cut the third set of relations down to a generating set for K.)

In particular, if G, H and K are finitely presented then so is $G *_K H$.

For any group L, homomorphisms $G *_K H \to L$ correspond to pairs of homomorphisms $(G \xrightarrow{h} L, H \xrightarrow{k} L)$ such that the square
$$\begin{array}{ccc} K & \xrightarrow{f} & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ H & \xrightarrow{k} & L \end{array}$$
commutes.

The Seifert–Van Kampen theorem says that in suitable cases the functor Π_1 preserves commutative squares having the universal property.

Suppose we are given a space X with subspaces A, B such that $A \cap B \neq \emptyset$. Choose a basepoint $x \in A \cap B$.

Then the inclusions
$$\begin{array}{ccc} A \cap B & \xrightarrow{i_1} & A \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ B & \xrightarrow{j_2} & X \end{array}$$
induce a commuting square
$$\begin{array}{ccc} \Pi_1(A \cap B, x) & \xrightarrow{i_{1*}} & \Pi_1(A, x) \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle j_{1*}} \\ \Pi_1(B, x) & \xrightarrow{j_{2*}} & \Pi_1(X, x) \end{array}$$
of group homomorphisms, and hence a homomorphism $\Pi_1(A, x) *_{\Pi_1(A\cap B, x)} \Pi_1(B, x) \to \Pi_1(X, x)$.

We'll call (X, A, B) a **Van Kampen triad** if this latter homomorphism is an isomorphism.

3.4 Theorem (Seifert–Van Kampen, open version). Suppose A and B are open in $X = A \cup B$ and $A \cap B$ is path-connected. Then (X, A, B) is a Van Kampen triad.

Proof. Let $u : I \to X$ be a loop based at x. Then $\{u^{-1}(A), u^{-1}(B)\}$ is an open covering of I. Let $\frac{1}{n}$ be a Lebesgue number for this cover, and subdivide I into the intervals $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, $1 \leqslant i \leqslant n$.

Let $u_i(t) = u\left(\frac{i-1}{n} + \frac{t}{n}\right)$, so that u_i is then a path in either A or B, for each i. Let $i_1, \ldots, i_k$ be the indices in $\{1, \ldots, n-1\}$ such that $u(\frac{i}{n}) \in A \cap B$. Since $A \cap B$ is path-connected, we can choose for each i_j a path v_j from x to $u\left(\frac{i_j}{n}\right)$ in $A \cap B$.

Now consider the product $(u_1 \cdot u_2 \cdots u_{i_1} \cdot \overline{v}_1)(v_1 \cdot u_{i_1+1} \cdots u_{i_2} \cdot \overline{v}_2)(v_2 \cdots) \cdots (\cdots u_n)$.

For $i_j + 1 \leqslant i \leqslant i_{j+1}$, the paths u_i all lie in the same one of A and B, since all the points $u(\frac{i}{n})$ for i in this range lie in either $A \setminus B$ or $B \setminus A$. So each of the bracketed sub-products is either a loop in A based at x, or a loop in B based at x.

$$\begin{aligned}\text{And}\quad [u] &= [u_1 \cdot u_2 \cdots u_n] \\ &= [(u_1 \cdot u_2 \cdots u_{i_1} \cdot \overline{v}_1)(v_1 \cdot u_{i_1+1} \cdots u_{i_2} \cdot \overline{v}_2) \cdots] \\ &= [u_1 \cdot u_2 \cdots u_{i_1} \cdot \overline{v}_1][v_1 \cdot u_{i_1+1} \cdots u_{i_2} \cdot \overline{v}_2] \cdots [v_{i_k} \cdots u_n]\end{aligned}$$

so every element of $\Pi_1(X, x)$ is expressible as a finite product of elements in the images of j_{1*} or j_{2*}.

So the canonical homomorphism $\Pi_1(A, x) *_{\Pi_1(A \cap B, x)} \Pi_1(B, x) \to \Pi_1(X, x)$ is surjective.

The representation of $[u]$ as a product, given u, is subject to the fact that we can:

(a) compose adjacent loops if they both lie in A or in B;

(b) subdivide an individual loop in A (resp. B) as a product of loops in A (resp. B);

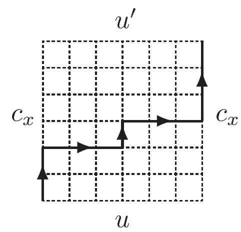
(c) regard a loop in $A \cap B$ as coming from A rather than B, or vice versa;

(d) replace consecutive terms $[w_1][w_2]$ by $[w_1 \cdot t][\overline{t} \cdot w_2]$, where t is a loop in $A \cap B$;

so as an element of $\Pi_1(A, x) *_{\Pi_1(A \cap B, x)} \Pi_1(B, x)$, it is unique.

Finally, suppose that $u \simeq u'$ rel $\{0, 1\}$. Let $H : I \times I \to X$ be a homotopy between them. Choose a Lebesgue number $\frac{1}{N}$ for the covering of $I \times I$ by open sets $\{H^{-1}(A), H^{-1}(B)\}$, where N is a multiple of the n's used in subdividing u and u'.

Subdivide $I \times I$ into squares of side $\frac{1}{N}$, and consider a sequence of paths from $(0, 0)$ to $(1, 1)$ as shown.



The images of these paths under H are loops in X based at x, each of which has a representation of the form that we considered, as such that any two consecutive loops differ by changing a single term by a homotopy in either A or B.

So they all correspond to the same element of $\Pi_1(A, x) *_{\Pi_1(A \cap B, x)} \Pi_1(B, x) \to \Pi_1(X, x)$. But the first corresponds to the element we chose as a representation of u and the last to the element we chose as a representation of u'. □

3.5 Examples.

(a) Let $n \geqslant 2$. Consider S^n as $A \cup B$, where $A = S^n \setminus \{x\}$ and $B = S^n \setminus \{-x\}$.
Each of A and B is homeomorphic to $\mathbb{R}^n$ (by stereographic projection), so in particular $\Pi_1(A) = \Pi_1(B) = \{e\}$. And $A \cap B \cong S^{n-1}$, so it's path-connected.
Hence $\Pi_1(S^n) \cong \{e\} *_? \{e\}$, which is trivial (so we don't need to take the amalgamation into account).
Note that this doesn't work for $n = 1$ since S^0 isn't path-connected.

(b) The n^{th} **Lens space** L_n is the quotient of the unit disc $B^2 \subset \mathbb{C}$ by the equivalence relation which identifies points z_1, z_2 of S^1 if $z_1^n = z_2^n$.
Let $A = L_n \setminus \{0\}$ and $B = \{z \in \mathbb{C} : |z| < 1\}$.
Then B is contractible and $A \simeq S^1$ (by projection outwards to the boundary), but $i_{1*} : \Pi_1(A \cap B) \to \Pi_1(A)$ is multiplication by n.
So our square of groups becomes $\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \end{array}$ and hence $\Pi_1(L_n) \cong \mathbb{Z}/n\mathbb{Z}$.
Note: $L_2 \cong \mathbb{RP}^2$, so this is another proof that $\Pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.

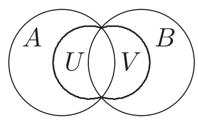
We would like to have a version of Seifert–Van Kampen for closed subspaces. For example, if X is a wedge union $A \vee B$, we'd like to take A and B as our subspaces. However, a path in $A \cup B$ is not in general decomposable as a finite product of paths in A or in B

In order to get around this, we need to assume that $A \cap B$ is 'nicely' embedded in A and in B, in the sense that it has an open neighbourhood in each that is topologically not very different from itself.

3.6 Definition. We say a subspace A of X is a **neighbourhood deformation retract** (NDR) if there exists an open $U \subset X$ with $A \subset U$, a continuous map $r : U \to A$ such that $r(x) = x$ for all $x \in A$, and a homotopy between id_U and r relative to A.

3.7 Theorem (Seifert–Van Kampen, closed version). Suppose A and B are closed subspaces of $X = A \cup B$ and that $A \cap B$ is path-connected and is a NDR in both A and B. Then (X, A, B) is a Van Kampen triad.

Proof. Let U and V be suitable open neighbourhoods of $A \cap B$ in A and B, respectively. Then $A \cup V$ is open in X, since its complement $B \setminus V$ is closed in B and hence in X. Moreover, the retractions $U \to A \cap B$ and $V \to A \cap B$ can be patched together to give a retraction $U \cup V \to A \cap B$ homotopic to $\mathrm{id}_{U \cup V}$.



We can patch the retraction $V \to A \cap B$ together with id_A to get a retraction $A \cup V \to A$ homotopic to id_A, and we similarly have a retraction $U \cup B \to B$ homotopic to id_B.

Now, by **3.4** we have $\Pi_1(X, x) \cong \Pi_1(A \cup V, x) *_{\Pi_1(U \cup V, x)} \Pi_1(U \cup B, x)$ for any $x \in U \cup V$ (so in particular for $x \in A \cap B$).

But we have a commutative diagram of fundamental groups:

$$\begin{array}{ccccc} \Pi_1(A, x) & \longleftarrow & \Pi_1(A \cap B, x) & \longrightarrow & \Pi_1(B, x) \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_1(A \cup V, x) & \longleftarrow & \Pi_1(U \cup V, x) & \longrightarrow & \Pi_1(U \cup B, x) \end{array}$$

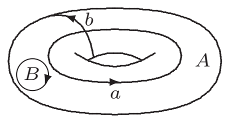
in which the vertical maps are all isomorphisms.

Hence $\Pi_1(A, x) *_{\Pi_1(A \cap B, x)} \Pi_1(B, x) \cong \Pi_1(A \cup V, x) *_{\Pi_1(U \cup V, x)} \Pi_1(U \cup B, x)$, and the result follows. $\square$

3.8 Examples.

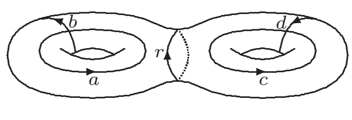
(a) $\Pi_1(S^1 \vee S^1)$ can be computed by taking A and B to be the two copies of S^1, with $A \cap B = \{x\}$. So $\Pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \cong F\{a, b\}$, the free group on two generators.

(b) Consider the torus T as $A \cup B$ where B is a closed disc B^2 and A is the closure of $T \setminus B$, so that $A \cap B \cong S^1$. Then $\Pi_1(B) = \{e\}$, and by question 5 on example sheet 1, we have $A \sim S^1 \vee S^1$, so $\Pi_1(A) \cong \langle a, b \rangle$.



The inclusion $A \cap B \to A$ sends a generator of $\Pi_1(A \cap B) \cong \mathbb{Z}$ to $aba^{-1}b^{-1} = [a, b]$ (the commutator of two groups elements). So our diagram of group homomorphisms becomes $\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \langle a, b \rangle \\ \downarrow & & \\ \{e\} & & \end{array}$ and the free product with amalgamation is $\langle a, b \,|\, aba^{-1}b^{-1} = e \rangle$, and this is the free abelian group of two generators, $\mathbb{Z} \times \mathbb{Z}$.

(c) Consider the double torus M_2. By cutting along a circle in the middle as shown, we can express M_2 as $A \cup B$, where A and B are homeomorphic to $T \setminus \{\text{open disc}\}$, and $A \cap B \cong S^1$.

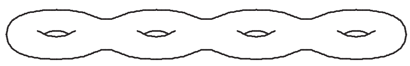


We get a diagram of the form
$$\begin{tikzcd} \Pi_1(A \cap B) \cong \langle r \rangle \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \Pi_1(A) \cong \langle a, b \rangle \\ \Pi_1(B) \cong \langle c, d \rangle & \end{tikzcd}$$

We have $i_{1*}(r) = aba^{-1}b^{-1}$ and $i_{2*}(r) = (cdc^{-1}d^{-1})^{-1}$. (Choose orientations of the loops such that this holds.)

So $\Pi_1(M_2) \cong \langle a, b, c, d : [a, b][c, d] = e \rangle$.

Similarly, we may consider the orientable surface of genus g, i.e. the g-holed torus. It can be shown by similar methods that



$$\Pi_1(M_g) \cong \langle a_1, b_1, a_2, b_2, \ldots a_g, b_g : [a_1, b_1][a_2, b_2]\ldots[a_g, b_g] = e \rangle.$$

What if we want to compute $\Pi_1\big(\bigvee_{\alpha \in A} X_\alpha\big)$ for an infinite wedge union of spaces X_α? The free product $*_{\alpha \in A} G_\alpha$ of an infinite family of groups G_α is defined to be the set of all formal finite products $g_1 \ldots g_r$ with $g_i \in \bigcup_{\alpha \in A} G_\alpha$, modulo the relations which say that we can replace $g_i g_{i+1}$ by h if $g_i, g_{i+1} \in G_\alpha$ for some α, and $g_i g_{i+1} = h$ holds in G_α, and we can delete identity elements.

Let $X = \bigvee_{\alpha \in A} X_\alpha$ be an infinite wedge union of spaces, i.e. the quotient of the disjoint union $\coprod_{\alpha \in A} X_\alpha$ by identifying a single point in each X_α with the basepoint x.

Note that the subset $F \subset X$ is closed iff $F \cap X_\alpha$ is closed in X_α for each α.

Warning. The Hawaiian earring $H = \bigcup_{i=1}^{\infty} C_n$ of **2.10** is not an infinite wedge union of circles: the set $\{(0, -\frac{2}{n}) : n \geqslant 1\}$ meets each C_n in a closed set, but isn't closed in H.

3.9 Proposition (Seifert–Van Kampen for infinite wedge unions). Suppose $\{x\}$ is a NDR in each X_α. Then $\Pi_1\big(\bigvee_{\alpha \in A} X_\alpha\big) \cong *_{\alpha \in A} \Pi_1(X_\alpha, x)$.

Proof. Let U_α be an open neighbourhood of x in X_α which deformation retracts onto x. Let $V_\alpha = X_\alpha \cup \big(\bigcup_{\beta \neq \alpha} U_\beta\big)$. Then V_α is open in X, and deformation retracts onto X_α.

Given a loop u in X based at x, consider the open covering $\{u^{-1}(V_\alpha) : \alpha \in A\}$ of I. Let $\frac{1}{n}$ be a Lebesgue number for this covering, then each $\big[\frac{i-1}{n}, \frac{i}{n}\big]$ is mapped by u into a single V_α. Applying the appropriate homotopy to each $u|_{[\frac{i-1}{n}, \frac{i}{n}]}$, we get a loop $r : I \to X$, homotopic to u rel $\{0, 1\}$, such that $r|_{[\frac{i-1}{n}, \frac{i}{n}]}$ maps into a single X_α.

By composing these subpaths as necessary, we get an expression for v as a finite product of loops, each lying in a single X_α. Thus any loop in $\bigvee_{\alpha \in A} X_\alpha$ is homotopic to one lying in $\bigvee_{\alpha \in A'} X_\alpha$ for some finite $A' \subset A$.

The rest of the proof is similar to **3.4**. □

Note in particular that if each X_α is homeomorphic to S^1, we see that $\Pi_1\big(\bigvee_{\alpha \in A} X_\alpha\big) \cong *_{\alpha \in A} \mathbb{Z} \cong \langle A \rangle$ is the free group generated by A.

Interlude : how are we doing so far?

We can now show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geqslant 2$, as follows.

If we are given a homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$, then we can restrict it to a homeomorphism $\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$. But $\mathbb{R}^n \setminus \{x\} \simeq S^{n-1}$, and we know $S^1 \not\simeq S^{n-1}$ for any $n > 2$, since $\Pi_1(S^1) \not\cong \Pi_1(S^{n-1})$.

But we can't yet prove that $\mathbb{R}^3$ and $\mathbb{R}^4$ are not homeomorphic.

Similarly, we can show that there is no deformation retraction $B^2 \to S^1$, since the inclusion $S^1 \hookrightarrow B^2$ induces a non-injective homomorphism $\Pi_1(S^1) \to \Pi_1(B^2)$.

But we can't solve this problem in higher dimensions. So we need functors giving us 'higher-dimensional information', in particular which distinguish between S^m and S^n for $m \neq n$.

There are higher-dimensional analogues Π_n of Π_1 which do this, but they are very difficult to calculate for non-trivial spaces.

Instead, we use the **homology groups** H_n, $n \geqslant 0$. Specifically, we'll consider **simplicial homology**, which is defined only for a restricted class of spaces called polyhedra, but which has the advantage of being (relatively) easy to calculate, once we've defined it.

Chapter 4 : Simplicial Complexes and Polyhedra

4.1 Definition. By an **affine subspace** of $\mathbb{R}^n$ we mean a coset of a vector subspace. The **dimension** of an affine subspace $A = V + x$ is the dimension of V.

Given $m+1$ vectors $x_0, x_1, \ldots, x_m$ in $\mathbb{R}^n$, the smallest affine subspace containing $\{x_0, \ldots, x_m\}$ is the set of all **affine linear combinations** $\sum_{i=0}^m t_i x_i$ where $\sum_{i=0}^m t_i = 1$.

Any vector of this form can be written as $x_0 + \sum_{i=1}^m t_i(x_i - x_0)$, so it lies in the coset of the space spanned by $\{x_1 - x_0, \ldots, x_m - x_0\}$ determined by x_0.

We say $x_0, \ldots, x_m$ are **affinely independent** if the vectors $x_1 - x_0, \ldots, x_m - x_0$ are linearly independent; equivalently if $\sum_{i=0}^m t_i x_i = 0$ and $\sum_{i=0}^m t_i = 0$ implies $t_i = 0$ for all i.

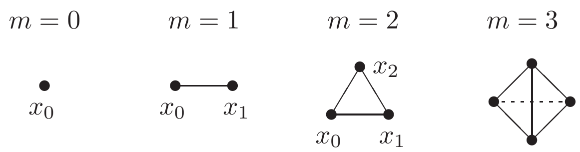
4.2 Definition.

(a) We say an affine linear combination $\sum_{i=0}^m t_i$, where $\sum_{i=0}^m t_i = 1$, is **convex** if $t_i \geqslant 0$ for each i.

(b) By an m**-simplex** in $\mathbb{R}^n$ we mean the subset

$$\langle x_0, x_1, \ldots, x_m \rangle = \left\{ \sum_{i=0}^m t_i x_i : \sum_{i=0}^m t_i = 1, \text{ and } t_i \geqslant 0 \text{ for each } i \right\}$$

of all convex affine combinations of a set of $m+1$ affinely independent points.



(c) The points $x_i = x_0, \ldots, x_m$ are called the **vertices** of the simplex $\sigma = \langle x_0, \ldots, x_m \rangle$.

By a **face** of a simplex $\langle x_0, \ldots, x_m \rangle$ we mean the span by a subset of $\{x_0, \ldots, x_m\}$. We write $\tau \leqslant \sigma$ to mean that τ is a face of σ. (Note that a face is also a simplex.)

The **boundary** $\partial\sigma$ of the simplex σ is the union of its proper faces.

The **interior** σ° is $\sigma \setminus \partial\sigma$. (So, if v is a vertex then $v^\circ = v$.)

Note: $\sum t_i x_i \in \partial\sigma \Leftrightarrow t_i = 0$ for at least one i, and $\sum t_i x_i \in \sigma^\circ \Leftrightarrow t_i > 0$ for all i.

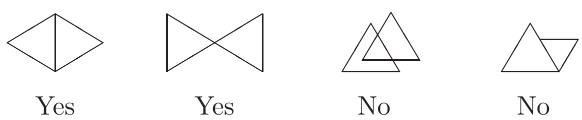
Warning. If σ is an m-simplex in $\mathbb{R}^n$, then σ° doesn't coincide with the interior of σ as a subset of $\mathbb{R}^n$ unless $m = n$. (If $m < n$ then the interior of σ in this sense is $\emptyset$.)

4.3 Definition. A **(geometric) simplicial complex** in $\mathbb{R}^n$ is a finite set K of simplices in $\mathbb{R}^n$ such that

(a) If $\sigma \in K$ and $\tau \leqslant \sigma$ then $\tau \in K$.

(b) If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is either $\emptyset$ or a face of both σ and τ.

Examples.



If K is a simplicial complex, the **polyhedron** $|K|$ of K is the union of all of the simplices in K, topologised as a subspace of $\mathbb{R}^n$. Note that $|K|$ is closed and bounded, hence compact.

A **triangulation** of a space X is a simplicial complex K together with a homeomorphism $|K| \to X$.

E.g., let $\triangle^n$ be the complex consisting of a single n-simplex with with all of its faces. Then $|\triangle^n|$ is an n-simplex, and it's homeomorphic to a closed ball $B^n \subset \mathbb{R}^n$, so we can think of $\triangle$ as a triangulation of B^n. Similarly, the subcomplex consisting of all proper faces of an n-simplex is a triangulation of S^{n-1}.

Given a complex K and an integer $d \geqslant 0$, the **d-skeleton** $K_{(d)}$ of K is the subcomplex consisting of all simplices in K of dimension $\leqslant d$.

So we can say that $\triangle^n_{(n-1)}$ is a triangulation of S^{n-1}. Note that we could also triangulate S^2 as an octahedron or an icosahedron.

4.4 Lemma. Let K be a simplicial complex. Then every point $x \in |K|$ lies in σ° for a unique $\sigma \in K$.

Proof. If $x \in |K|$ then $x \in \sigma = \langle v_0, v_1, \ldots, v_m \rangle$ for some $\sigma \in K$, so $x = \sum_{i=0}^m t_i v_i$ for some t_i with $t_i \geqslant 0$, $\sum_{i=0}^m t_i = 1$.

Then $x \in \tau^\circ$ for τ the face of σ spanned by the v_i for which $t_i > 0$.

If $x \in \sigma^\circ \cap \tau^\circ$ then $\sigma \cap \tau = \sigma = \tau$, since $\sigma \cap \tau$ contains points in σ° and τ°. □

4.5 Definition. An **abstract simplicial complex** is a pair (V, K) where V is a finite set (whose elements are called **vertices**) and K is a set of non-empty finite subsets of V, called **(abstract) simplices**, such that

(a) If $\sigma \in K$ and $\emptyset \neq \tau \subset \sigma$ then $\tau \in K$

(b) If $v \in V$ then $\{v\} \in K$.

A geometric simplicial complex K gives rise to an abstract complex, where we take $V = \{0\text{-simplices in } K\}$ and identify each simplex in K with its set of vertices.

By a **geometric realisation** of an abstract subspace (V, K) we mean a geometric complex $\overline{K}$ whose underlying abstract complex is isomorphic to (V, K).

4.6 Lemma. Every abstract complex has a geometric realisation.

Proof. Suppose V has $n+1$ elements. Identify these with the vertices of an n-simplex $\triangle^n$, and then every non-empty finite subset of V defines a face of $\triangle^n$. The faces corresponding to sets in K define a subcomplex of $\triangle^n$ whose abstract complex is (isomorphic to) (V, K). □

4.7 Definition. Let K and L be simplicial complexes. A **simplicial map** $f : K \to L$ is a mapping $\{\text{vertices of } K\} \to \{\text{vertices of } L\}$ such that if $\sigma = \langle v_0, v_1, \ldots, v_m \rangle$ is a simplex of K, then the set $\langle f(v_0), f(v_1), \ldots, f(v_m) \rangle$ spans a simplex $f[\sigma]$ of L.

(Note that $f[\sigma]$ might have lower dimension than σ.)

4.8 Lemma. A simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$. Moreover, if $g : L \to M$ is another simplicial map, then $|g|\,|f| = |gf| : |K| \to |M|$.

Proof. For $\sigma = \langle v_0, v_1, \ldots, v_m \rangle$, we define the restriction of $|f|$ to σ, written f_σ, to be the mapping $\sum_{i=0}^m t_i v_i \mapsto \sum_{i=0}^m t_i f(v_i)$, which is a continuous map $\sigma \to f[\sigma]$.

If $\tau \leqslant \sigma$ then clearly $f_\tau = f_\sigma|_\tau$. Hence if σ and τ are any two simplices of K, we have $f_\sigma|_{\sigma\cap\tau} = f_{\sigma\cap\tau} = f_\tau|_{\sigma\cap\tau}$. So we can patch the f_σ together to get a continuous map $|f| : |K| \to |L|$.

The second statement is immediate from the form of the definition. $\square$

4.9 Corollary. Any two geometric realisations of a given abstract complex have homeomorphic polyhedra.

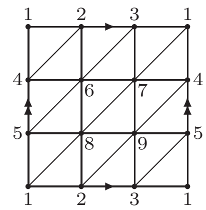
Proof. Suppose K and L have the same underlying abstract complex M. The identity map id_M can be regarded as a simplicial map $K \to L$ or $L \to K$, so it induces continuous maps $|K| \to |L|$ and $|L| \to |K|$ whose composite either way around is the identity. $\square$

To define a triangulation of a given space X, it's enough to decompose X into pieces homeomorphic to simplices and show that they fit together according to the rules in **4.3**.

Example. Consider the torus $T = S^1 \times S^1$ as the quotient of the unit square with points on opposite edges identified.

We can triangulate it as shown, using nine vertices and eighteen 2-simplices.

This is not the best possible: T can be triangulated with seven vertices and fourteen 2-simplices.



Not every continuous map $|K| \to |L|$ is induced by simplicial maps $K \to L$. But it turns out that every such map is homotopic to one induced by a simplicial map $K^{(r)} \to L$ for some r (where $K^{(r)}$ will be defined shortly).

4.10 Definition. Let K be a simplicial complex, and $x \in |K|$.

(a) The **star** of x in K is the subset $st_K(x) = \bigcup\{\sigma^\circ : \sigma \in K, x \in \sigma\}$ of K.
Note that $st_K(x)$ is open in $|K|$, since $|K| \backslash st_K(x)$ is $|L|$, where L is the subcomplex $\{\tau \in K : x \notin \tau\}$ of K.
Also, $\{st_K(v) : v \text{ is a vertex of } K\}$ covers $|K|$, by **4.4**.

(b) The **link** of x in K is $lk_K(x) = \bigcup\{\tau \in K : x \notin \tau, \text{ and } \exists\, \sigma \in K \text{ such that } x \in \sigma \text{ and } \tau \leqslant \sigma\}$.
Note that $lk_K(x)$ is closed in $|K|$, since it's the polyhedron of a subcomplex.

4.11 Definition. Let K, L be simplicial complexes, and $f : |K| \to |L|$ a continuous function. By a **simplicial approximation** to f, we mean a mapping $g : \{\text{vertices of } K\} \to \{\text{vertices of } L\}$ such that for each v, $f(st_K(v)) \subset st_L(g(v))$.

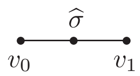
4.12 Lemma. If g is a simplicial approximation to the continuous function f, then g is a simplicial map and $|g| \simeq f$.

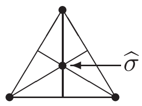
Proof. Let $\sigma \in K$. For any $x \in \sigma^\circ$, we have $x \in st_K(v)$ for all vertices v of σ, so $f(x) \in st_L(g(v))$ for all such v. Hence if τ is the simplex of L such that $f(x) \in \tau^\circ$, then all the $g(v)$ are vertices of τ, so they span a simplex which is a face of τ.

Also, for any $x \in |K|$, if $f(x) \in \tau^\circ$ then $|g|(x) \in \tau$, so the segment joining $f(x)$ to $|g|(x)$ lies in τ. So the linear homotopy given by $H(x,t) = tf(x) + (1-t)|g|(x)$ takes values in $|L|$. $\square$

4.13 Definition.

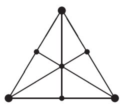
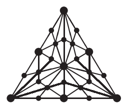
(a) Let $\sigma = \langle v_0, \ldots, v_m \rangle$. The **barycentre** of σ is the point $\widehat{\sigma} = \frac{1}{m+1} \sum_{i=0}^{m} v_i$.





(b) The **(first) barycentric subdivision** of a simplicial complex K is the simplicial complex K' whose vertices are the barycentres of all the subcomplexes of K, and in which $\{\widehat{\sigma}_0, \widehat{\sigma}_1, \ldots, \widehat{\sigma}_m\}$ spans a simplex if and only if the σ_i can be ordered so that $\sigma_0 < \sigma_1 < \ldots < \sigma_m$.

The r^{th} barycentric subdivision of K is defined inductively by $K^{(r)} = (K^{(r-1)})'$, with $K^{(1)} = K'$.

E.g., if $K = \triangle^2$ then K' is:  and $K^{(2)}$ is: 

4.14 Proposition. K' is a simplicial complex and $|K'| = |K|$.

Proof. First, if $\sigma_0 < \sigma_1 < \ldots < \sigma_m$, then $\widehat{\sigma}_0, \widehat{\sigma}_1, \ldots, \widehat{\sigma}_m$ are affinely independent.

If we had a non-trivial affine relation $\sum_{i=0}^{m} t_i \widehat{\sigma}_i = 0$, then by considering the last non-zero t_i, we'd get an expression for $\widehat{\sigma}_i$ as a convex affine combination of $\widehat{\sigma}_0, \ldots, \widehat{\sigma}_{i-1}$, but the latter all lie in the proper face σ_{i-1} of σ_i, so this is impossible.

Next, we show $|K'| = |K|$. The inclusion $|K'| \subset |K|$ is clear since each $\langle \widehat{\sigma}_0, \ldots, \widehat{\sigma}_m \rangle$ in K' is contained in the single simplex σ_m of K (having ordered $\sigma_0 < \ldots < \sigma_m$).

Suppose $x \in \langle v_0, \ldots, v_m \rangle \in K$. We may assume that the v_i have been ordered so that $x = \sum_{i=0}^{m} t_i v_i$ with $t_0 \geqslant t_1 \geqslant \ldots \geqslant t_m$. Then

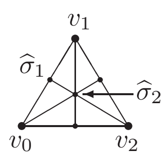
$$\begin{aligned} x &= (t_0 - t_1)v_0 + (t_1 - t_2)(v_0 + v_1) + (t_2 - t_3)(v_0 + v_1 + v_2) \\ &\quad + \ldots + t_m(v_0 + \ldots + v_m) \\ &= (t_0 - t_1)\langle v_0 \rangle\widehat{\ } + 2(t_1 - t_2)\langle v_0, v_1 \rangle\widehat{\ } + 3(t_2 - t_3)\langle v_0, v_1, v_2 \rangle\widehat{\ } \\ &\quad + \ldots + (m+1)t_m \langle v_0, \ldots, v_m \rangle\widehat{\ } \end{aligned}$$

where, for example, by $\langle v_0, v_1 \rangle\widehat{\ }$ we mean $\widehat{\sigma}$ for the simplex $\sigma = \langle v_0, v_1 \rangle$. And this shows that x belongs to the simplex $\big\langle \langle v_0 \rangle\widehat{\ }, \langle v_0, v_1 \rangle\widehat{\ }, \ldots, \langle v_0, \ldots, v_m \rangle\widehat{\ } \big\rangle$ of K'.

Now let $\sigma' = \langle \widehat{\sigma}_0, \ldots, \widehat{\sigma}_m \rangle$ and $\tau' = \langle \widehat{\tau}_0, \ldots, \widehat{\tau}_n \rangle$ be simplices of K', with $\sigma_0 < \ldots < \sigma_m$ and $\tau_0 < \ldots < \tau_n$.

Then $\sigma' \cap \tau' \subset \sigma_m \cap \tau_n$, which is a simplex π of K. Now $\sigma' \cap \pi$ is a face $\langle \widehat{\sigma}_0, \ldots, \widehat{\sigma}_r \rangle$ of σ', and $\tau' \cap \pi$ is a face of $\langle \widehat{\tau}_0, \ldots, \widehat{\tau}_s \rangle$ of τ', where σ_r (resp. τ_s) is the last member of the sequence $\sigma_0, \ldots, \sigma_m$ (resp. $\tau_0, \ldots, \tau_n$) contained in π. So we've reduced to the case when σ' and τ' are contained in a single simplex π of K.

Suppose, for example, that $x \in \langle v_0, \widehat{\sigma}_1, \widehat{\sigma}_2 \rangle \cap \langle v_1, \widehat{\sigma}_1, \widehat{\sigma}_2 \rangle$, where v_0 and v_1 are distinct vertices of the 1-simplex σ_1. Then in the expression of x as $\sum_{i=0}^{2} t_i v_i$ we have both $t_0 \geqslant t_1$ and $t_1 \geqslant t_0$. Hence x lies in the simplex $\langle \widehat{\sigma}_1, \widehat{\sigma}_2 \rangle$ spanned by the common vertices of the two simplices. $\square$



4.15 Remark. Let G be any function mapping each barycentre $\widehat{\sigma}$ to one of the vertices of σ. Then g is a simplicial approximation to $\mathrm{id}_{|K|} : |K'| \to |K|$, since we have $st_{K'}(\widehat{\sigma}) \subset st_K(\widehat{\sigma}) \subset st_K(v)$ for any vertex v of $\widehat{\sigma}$ (the latter since $\widehat{\sigma} \in \tau$ implies $\sigma \leqslant \tau$).

4.16 Definition. We define the **diameter** of a bounded subset A of a metric space to be $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$.

Note that if $A = \sigma$ is a simplex in $\mathbb{R}^n$, this supremum is attained at a pair of vertices of σ, so $\operatorname{diam}\sigma$ is the length of the longest 1-dimensional face of σ. For a simplicial complex K, we define the **mesh** of K to be $\operatorname{mesh} K = \max\{\operatorname{diam}\sigma : \sigma \in K\}$.

4.17 Lemma. Suppose that K is a simplicial complex in which all simplices have diameter at most $\frac{1}{n}$. Then $\operatorname{mesh} K' \leqslant \frac{n}{n+1} \operatorname{mesh} K$.

Proof. Let $\langle \widehat{\tau}, \widehat{\sigma} \rangle$ be a 1-simplex of K', where $\tau < \sigma$. Then $\|\widehat{\sigma} - \widehat{\tau}\| \leqslant \max\{\|\widehat{\sigma} - v\| : v \text{ a vertex of } \sigma\}$.

Let $\sigma = \langle v_0, v_1, \ldots, v_m \rangle$ and assume that the maximum is attained at v_0.

Then $\|\widehat{\sigma} - v_0\| = \left\| \frac{1}{m+1} \sum_{i=1}^{m} (v_i - v_0) \right\| \leqslant \frac{1}{m+1} \sum_{i=0}^{m} \|v_i - v_0\| \leqslant \frac{m}{m+1} \operatorname{diam}\sigma$.

But $m \leqslant n$, so $\frac{m}{m+1} \operatorname{diam}\sigma \leqslant \frac{n}{n+1} \operatorname{mesh} K$.

Hence $\operatorname{mesh} K^{(r)} \leqslant \left(\frac{n}{n+1}\right)^r \operatorname{mesh} K$. In particular, $\operatorname{mesh} K^{(r)} \to 0$ as $r \to \infty$. □

4.18 Theorem (Simplicial Approximation theorem). Let K and L be simplicial complexes, and $f : |K| \to |L|$ a continuous map. Then for sufficiently large r there is a simplicial map $g : K^{(r)} \to L$ which is a simplicial approximation to f.

Proof. Consider the open covering $\left\{ f^{-1}(st_L(w)) : w \text{ a vertex of } L \right\}$ of $|K|$. Since $|K|$ is a compact metric space, we can choose r so that $\operatorname{mesh} K^{(r)}$ is a Lebesgue number for this covering.

Now, for any vertex v of $K^{(r)}$, we have $st_{K^{(r)}}(v) \subset B\big(v, \operatorname{mesh} K^{(r)}\big) \subset f^{-1}(st_L(w))$ for some w, so if we choose such a w and call it $g(v)$, then g is the required simplicial approximation. □

4.19 Remark. Suppose $f : |K| \to |L|$ is a continuous map and that $f|_{|M|}$ is simplicial for some subcomplex M of K. Then we can choose our simplicial approximation g to coincide with f at all vertices of M, and the homotopy $|g| \simeq f$ will fix these points.

Chapter 5 : Chains and Homology

5.1 Definition. By an **orientation** of a simplex σ, we mean an equivalence class of total orderings of the vertices of σ, where we regard two orderings as equivalent if they differ by an even permutation. (The 0-simplex, a point, is arbitrarily assigned a $+$ or $-$ sign.)

5.2 Definition. Let K be a simplicial complex. By an **n-chain** on K, we mean a formal finite sum $\sum_{i=1}^{r} m_i\sigma_i$, where $m_i \in \mathbb{Z}$ and the σ_i are all oriented n-simplices of K. We identify $-m\sigma$ with $m\overline{\sigma}$, where $\overline{\sigma}$ is σ with the opposite orientation.

The set of all n-chains on K forms an abelian group $C_n(K)$, isomorphic to $\mathbb{Z}^{k_n}$ where k_n is the number of n-simplices in K.

5.3 Definition. Let $\sigma = \langle v_0, \ldots, v_n\rangle$ be an oriented n-simplex. We define the **(algebraic) boundary** to be the $(n-1)$-chain $d(\sigma) = \sum_{i=0}^{n}(-1)^i\langle v_0, \ldots, \cancel{v_i}, \ldots, v_n\rangle$. (The algebraic boundary of the 0-simplex is 0.)

To check that this is well-defined, consider the effect of transposing the vertices v_{i-1} and v_i. This changes the orientation of each term in $d(\sigma)$. Since the symmetric group S_{n+1} is generated by transpositions of adjacent pairs, this is sufficient.

Hence we can extend d to a homomorphism $C_n(K) \to C_{n-1}(K)$ by setting $d(\sum m_i\sigma_i) = \sum m_i d(\sigma_i)$.

5.4 Lemma. The composite $C_n(K) \xrightarrow{d} C_{n-1}(K) \xrightarrow{d} C_{n-2}(K)$ is identically 0.

Proof. It's enough to show that $d^2(\sigma) = 0$ for an n-simplex σ. But each $(n-2)$-dimensional face $\langle v_0, \ldots, \cancel{v_i}, \ldots, \cancel{v_j}, \ldots, v_n\rangle$ of σ occurs twice in $d^2(\sigma)$, but with opposite sign: once with sign $(-1)^{i+j}$ and once with sign $(-1)^{i+j-1}$. $\square$

5.5 Definition.

(a) An **n-cycle** of K is an n-chain c such that $d(c) = 0$. We write $Z_n(K)$ for the subgroup of $C_n(K)$ consisting of cycles, i.e. the kernel of $d : C_n(K) \to C_{n-1}(K)$.

(b) We write $B_n(K)$ for the subgroup of $C_n(K)$ consisting of boundaries of $(n+1)$-chains, i.e. the image of $d : C_{n+1}(K) \to C_n(K)$.

(c) We define the $n^{\mathbf{th}}$ **homology group** $H_n(K)$ to be the quotient $Z_n(K)/B_n(K)$.

5.6 Proposition. Let $f : K \to L$ be a simplicial map. Then f induces homomorphisms $C_n(f) = f_n : C_n(K) \to C_n(L)$ such that

$$\begin{array}{ccc} C_n(K) & \xrightarrow{d} & C_{n-1}(K) \\ \downarrow f_n & & \downarrow f_{n-1} \\ C_n(L) & \xrightarrow{d} & C_{n-1}(L) \end{array}$$

commutes for all n. Hence f also induces homomorphisms $H_n(f) = f_* : H_n(K) \to H_n(L)$ for all n.

Moreover, if $g : L \to M$ is another simplicial map, then $C_n(g)C_n(f) = C_n(gf)$ and $H_n(g)H_n(f) = H_n(gf)$.

Proof. Let $\langle v_0, \ldots, v_n \rangle$ be an n-simplex of K. We define $f_n(\sigma) = \langle fv_0, \ldots, fv_n \rangle$ if the fv_i are all distinct, and 0 otherwise. The functoriality, i.e. $g_n f_n = (gf)_n$, is then obvious.

To verify $f_{n-1}d = df_n$, there are three cases, depending on $fv_0, \ldots, fv_n$.

(i) If they're all distinct, then $f_{n-1}d(\sigma) = df_n(\sigma) = \sum_{i=1}^{n}(-1)^i \langle fv_0, \ldots, \cancel{fv_i}, \ldots, fv_n \rangle$.

(ii) If they span a simplex of dimension $< n - 1$, then $f_{n-1}d(\sigma) = df_n(\sigma) = 0$.

(iii) If there is exactly one coincidence among them, say $fv_i = fv_j$, then $df_n(\sigma) = 0$ and $f_{n-1}d(\sigma) = (-1)^i \langle fv_0, \ldots, \cancel{fv_i}, \ldots, fv_n \rangle + (-1)^j \langle fv_0, \ldots, \cancel{fv_j}, \ldots, fv_n \rangle$. But these two simplices are the same, and their orientations differ by $(-1)^{i-j+1}$.

Hence f_n restricts to homomorphisms $Z_n(K) \to Z_n(L)$ and $B_n(K) \to B_n(L)$, and so it induces a well-defined homomorphism $H_n(K) \to H_n(L)$. $\square$

5.7 Definition. By a **(chain) complex**, we mean a sequence of abelian groups C_n $(n \in \mathbb{Z})$ and homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_n d_{n+1} = 0$. Write $C_\bullet$ for a chain complex, and define $Z_n(C_\bullet) = \ker d_n$, $B_n(C_\bullet) = \operatorname{im} d_{n+1}$ and $H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet)$. We usually just write d for each d_n.

A **chain map** $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ satisfying $f_{n-1}d = df_n$ for all n. Given chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, a **chain homotopy** $h_\bullet$ from $f_\bullet$ to $g_\bullet$ is a sequence of homomorphisms $h_n : C_n \to D_{n+1}$ such that $g_n - f_n = dh_n + h_{n-1}d$ for all n.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dl] & C_n \arrow[r, "d"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \arrow[dl] \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d"'] & D_n \arrow[r, "d"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

5.8 Lemma. Suppose there is a chain homotopy, $f_\bullet \simeq g_\bullet$ via $h_\bullet$. Then $f_\bullet$ and $g_\bullet$ induce the same homomorphisms $f_*, g_* : H_n(C_\bullet) \to H_n(D_\bullet)$.

Proof. Suppose $z \in Z_n(C_\bullet)$ represents $x \in H_n(C_\bullet)$. Then $g_n(z) - f_n(z) = dh_n(z) \in B_n(D_\bullet)$. So $g_n(z)$ and $f_n(z)$ represent the same element of $H_n(D_\bullet)$, so $f_*(x) = g_*(x)$. $\square$

Exercise. Verify that chain homotopy is an equivalence relation on chain maps, and that $f_\bullet \simeq g_\bullet$ implies $k_\bullet f_\bullet \simeq k_\bullet g_\bullet$ and $f_\bullet l_\bullet \simeq g_\bullet l_\bullet$ where the composites are defined.

5.9 Definition. We say a simplicial complex K is a **cone** if it has a vertex v_0 such that every simplex of K is a face of a simplex containing v_0. (Equivalently, $st_K(v_0)$ is dense in $|K|$, or $st_K(v_0) \cup lk_K(v_0) = |K|$.)

5.10 Proposition. Suppose K is a cone, with distinguished vertex v_0. Then the inclusion $i : \{v_0\} \to K$ induces a chain homotopy equivalence $C_\bullet(\{v_0\}) \to C_\bullet(K)$ and hence an isomorphism $H_n(\{v_0\}) \to H_n(K)$ for all n.

(We say that $f_\bullet : C_\bullet \to D_\bullet$ is a **chain homotopy equivalence** if there exists $g_\bullet : D_\bullet \to C_\bullet$ such that $g_\bullet f_\bullet \simeq \mathrm{id}_{C_\bullet}$ and $f_\bullet g_\bullet \simeq \mathrm{id}_{D_\bullet}$.)

Proof. There is a unique simplicial map $r : K \to \{v_0\}$ and $ri = \mathrm{id}_{\{v_0\}}$, so $r_n i_n = \mathrm{id}_{C_n\{v_0\}}$ for all n, and so $r_* i_* = \mathrm{id}_{H_n(\{v_0\})}$.

Define $h_n : C_n(K) \to C_{n+1}(K)$ by $h_n(\sigma) = \langle v_0, \sigma \rangle$ if $v_0 \notin \sigma$, and $h_n(\sigma) = 0$ if $v_0 \in \sigma$ (where $\langle v_0, \sigma \rangle$ means $\langle v_0, v_1, \ldots, v_{n+1} \rangle$, for $\sigma = \langle v_1, \ldots, v_{n+1} \rangle$).

Let $\sigma \in C_n$. If $n > 0$ and $v_0 \notin \sigma$ then $v_0 \notin \tau$ for any face τ of σ, so

$$\begin{aligned} dh_n(\sigma) + h_{n-1}d(\sigma) &= d(\langle v_0, \sigma\rangle) + \langle v_0, d(\sigma)\rangle \\ &= \sigma - \langle v_0, d(\sigma)\rangle + \langle v_0, d(\sigma)\rangle \\ &= \sigma \\ &= \sigma - i_n r_n(\sigma), \text{ since } r_n(\sigma) = 0. \end{aligned}$$

If $n > 0$ and $v_0 \in \sigma$, assume for simplicity that $\sigma = \langle v_0, \tau\rangle$, and then

$$\begin{aligned} dh_n(\sigma) + h_{n-1}d(\sigma) &= d(0) + h_{n-1}\big(\tau - \langle v_0, d(\tau)\rangle\big) \\ &= 0 + \langle v_0, \tau\rangle - 0 \\ &= \sigma \\ &= \sigma - i_n r_n(\sigma), \text{ as before.} \end{aligned}$$

If $n = 0$, we have $i_0 r_0(v) = v_0$ and $h_{n-1}d(v) = 0$ for all v. Then

$$\begin{aligned} dh_0(v) &= d\langle v_0, v\rangle = v - v_0 = v - i_0 r_0(v) \quad \text{if } v \neq v_0 \\ dh_0(v_0) &= 0 = v_0 - i_0 r_0(v_0) \end{aligned}$$

We've shown that $dh_n + h_{n-1}d = \mathrm{id}_{C_n(K)} - i_n r_n$ for all n. So by **5.8**, $i_* r_* = \mathrm{id}_{H_n(K)}$.

Hence i_* and r_* are isomorphisms, and $H_n(K) \cong H_n(\{v_0\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$ □

5.11 Corollary. Let $K = \triangle^n_{(n-1)}$ be the boundary of an n-simplex.

Then for $n \geqslant 2$, $H_m(K) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0 \text{ or } n-1 \\ 0 & \text{otherwise.} \end{cases}$

(Note that K triangulates S^{n-1}.)

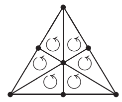
Proof. For $m \leqslant n-1$, $C_m(K) = C_m(\triangle^n)$, hence $Z_m(K) = Z_m(\triangle^n)$ and $B_{m-1}(K) = B_{m-1}(\triangle^n)$. So K has the same homology groups as $\triangle^n$ in dimensions $\leqslant n-2$. And $\triangle^n$ is a cone, so $H_0(K) = \mathbb{Z}$ and $H_m(K) = 0$ for all $0 < m \leqslant n-2$.

Now, $Z_{n-1}(K) = Z_{n-1}(\triangle^n)$ and $H_{n-1}(\triangle^n) = 0$, so $Z_{n-1}(K) = B_{n-1}(\triangle^n)$, and since $Z_n(\triangle^n) = 0$, the map d_n is injective, and hence $B_{n-1}(\triangle^n) \cong C_n(\triangle^n) \cong \mathbb{Z}$. Also, $B_{n-1}(K) = 0$ since K has no n-simplices, and hence $H_{n-1}(K) \cong \mathbb{Z}/0 \cong \mathbb{Z}$. □

5.12 Proposition. Let $a : K' \to K$ be a simplicial approximation to $\mathrm{id}_{|K|}$. Then the chain map $a_\bullet : C_\bullet(K') \to C_\bullet(K)$ is a chain homotopy equivalence.

Proof. To simplify the calculation, suppose we have chosen a total ordering of the vertices of K, and that $a(\widehat{\sigma})$ is the first vertex of σ for each simplex of K.

We define a chain map $s_\bullet : C_\bullet(K) \to C_\bullet(K')$ by induction on dimension. Let $s_0(v) = v$ for a vertex v of K, and for $n > 0$ let $s_n(\sigma) = \langle \widehat{\sigma}, s_{n-1}d(\sigma)\rangle$. (Exercise: check what this gives for a simplex.)



We need to verify that $s_\bullet$ is a chain map:

$$\begin{aligned} ds_n(\sigma) &= s_{n-1}d(\sigma) - \langle \widehat{\sigma}, ds_{n-1}d(\sigma)\rangle \\ &= s_{n-1}d(\sigma) - \langle \widehat{\sigma}, s_{n-2}d^2(\sigma)\rangle \quad \text{by induction hypothesis} \\ &= s_{n-1}d(\sigma) \end{aligned}$$

Let $L_\sigma \subset K'$ be the subcomplex formed by the simplices of K' which are contained in a given simplex σ of K. Note that L_σ is a cone with distinguished vertex $\widehat{\sigma}$, and that if $\sigma_1 \leqslant \sigma_2$ then $L_{\sigma_1} \subset L_{\sigma_2}$.

Note also that $s_n(\sigma)$ lies in the subgroup $C_n(L_\sigma)$ of $C_n(K')$.

For an n-simplex $\sigma = \langle v_0, v_1, \ldots, v_n\rangle$ of K, let $\sigma_i = \langle v_i, v_{i+1}, \ldots, v_n\rangle$. Then the n-simplex $\langle \widehat{\sigma}, \widehat{\sigma}_1, \ldots, \widehat{\sigma}_n\rangle$ is mapped to σ by a_n, and all other n-simplices in L_σ are mapped to 0 by a_n.

But σ_1 occurs with sign $+$ in $d(\sigma)$, so by induction hypothesis $\langle \widehat{\sigma}_1, \ldots, \widehat{\sigma}_n\rangle$ occurs with sign $+$ in $s_{n-1}d(\sigma)$, so $\langle \widehat{\sigma}, \widehat{\sigma}_1, \ldots, \widehat{\sigma}_n\rangle$ occurs with sign $+$ in $s_n(\sigma)$. Hence $a_n s_n(\sigma) = \sigma$ for all $\sigma \in K$.

Now we need a chain homotopy $\big(h_n : C_n(K') \to C_{n+1}(K')\big)_{n\geqslant 0}$ such that for all n we have $dh_n(\tau) + h_{n-1}d(\tau) = \tau - s_n a_n(\tau)$.

We'll demand also that $h_n(\tau) \in C_{n+1}(L_\sigma)$, where σ is the highest-dimensional simplex of K such that $\widehat{\sigma}$ is a vertex of τ. We define h_n by induction on n.

Let $h_0(\tau) = \begin{cases} 0 & \text{if } \tau = \widehat{\sigma} \text{ for a 0-simplex } \sigma \text{ of } K \\ \langle s_0 a_0 \tau, \tau\rangle & \text{if } \widehat{\tau} = \widehat{\sigma} \text{ for an } n\text{-simplex } \sigma \text{ of } K, n > 0 \end{cases}$

Now assume h_{n-1} has been defined, let τ be an n-simplex, and consider the n-chain $c = \tau - s_n a_n(\tau) - h_{n-1}d(\tau) \in C_n(L_\sigma)$.

$$\begin{aligned} d(c) &= d(\tau) - s_{n-1}a_{n-1}d(\tau) - dh_{n-1}d(\tau) \\ &= h_{n-2}d^2(\tau) \quad \text{by induction hypothesis} \\ &= 0 \quad \text{since } d^2 = 0 \end{aligned}$$

So $c \in Z_n(L_\sigma) = B_n(L_\sigma)$ by **5.10**. So we can choose $x \in C_{n+1}(L_\sigma)$ such that $d(x) = c$, and then define $h_n(\tau) = x$. □

It follows that if $f : |K| \to |L|$ is an arbitrary continuous map between polyhedra, then we can choose a simplicial approximation $g : K^{(r)} \to L$ to f for some $r \geqslant 0$, and then we have a family of homomorphisms $H_n(K) \xrightarrow{(s^r)_*} H_n(K^{(r)}) \xrightarrow{g_*} H_n(L)$.

We need to show that if $f, g : |K| \to |L|$ are homotopic (continuous) maps and we have simplicial approximations $\widetilde{f} : K^{(r_1)} \to L$ and $\widetilde{g} : K^{(r_2)} \to L$ to f and g respectively, then the composites

$$H_n(K) \xrightarrow{(s^{r_1})_*} H_n(K^{(r_1)}) \xrightarrow{\widetilde{f}_*} H_n(L) \quad \text{and} \quad H_n(K) \xrightarrow{(s^{r_2})_*} H_n(K^{(r_2)}) \xrightarrow{\widetilde{g}_*} H_n(L)$$

are equal.

We may assume that $r_1 = r_2$. For if $r_1 < r_2$ then the composite $K^{(r_2)} \xrightarrow{a^{r_2-r_1}} K^{(r_1)} \xrightarrow{\widetilde{f}} L$ is still a simplicial approximation to f, and the following triangle commutes:

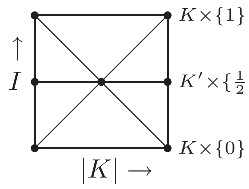
$$\begin{tikzcd} C_\bullet(K) \arrow[rr, "(s^{r_2})_\bullet"] \arrow[dr, "(s^{r_1})_\bullet"'] & & C_\bullet(K^{(r_2)}) \arrow[dl, "(a^{r_2-r_1})_\bullet"] \\ & C_\bullet(K^{(r_1)}) & \end{tikzcd}$$

Thus we are reduced to proving the following.

5.13 Proposition. Let $f, g : K \to L$ be simplicial maps such that $|f| \simeq |g|$. Then f and g induce the same homomorphisms $H_n(K) \to H_n(L)$ for each $n \geqslant 0$.

Proof. Let $H : |K| \times I \to |L|$ be a homotopy from $|f|$ to $|g|$. We triangulate the space $|K| \times I$ as follows. (See example sheet 3, question 5.)

As vertices, take all points $(v, 0)$ and $(v, 1)$ where v is a vertex of K, and all points $(\widehat{\sigma}, \frac{1}{2})$ where σ is a simplex of K.



For simplices, take: all simplices $\sigma \times \{0\}$ and $\sigma \times \{1\}$ where σ is a simplex of K; all simplices $\tau \times \{\frac{1}{2}\}$ where τ is a simplex of K'; and all simplices of the form $(\sigma \times \{0\}) \# (\tau \times \{\frac{1}{2}\})$ or $(\sigma \times \{1\}) \# (\tau \times \{\frac{1}{2}\})$ where σ is a simplex of K, τ is a simplex of K', and $\sigma \leqslant \sigma'$ for all vertices $\widehat{\sigma}'$ of τ.

(Here, $\sigma \# \tau$ means the simplex spanned by the union of the vertices of σ and τ, assuming that this union is an affinely independent set.)

It can be shown that this is indeed a triangulation of $|K| \times I$.

Call this simplicial complex M, and for each $\sigma \in K$ let M_σ denote the subcomplex of M formed by simplices contained in $\sigma \times I$. Note that M_σ is a cone, with distinguished vertex $(\widehat{\sigma}, \frac{1}{2})$.

We have simplicial maps $i_0, i_1 : K \to M$ defined by $v \mapsto (v, 0)$ and $v \mapsto (v, 1)$ respectively, and a simplicial map $r : M \to K$ defined by $r(v, 0) = r(v, 1) = v$ and $r(\widehat{\sigma}, \frac{1}{2}) = a(\widehat{\sigma})$ where $a : K' \to K$ is a simplicial approximation to $\mathrm{id}_{|K|}$.

Clearly $ri_0 = ri_1 = \mathrm{id}_K$. We claim that $i_0 r$ and $i_1 r$ induce chain maps $C_\bullet(M) \to C_\bullet(M)$ which are chain-homotopic to the identity. To construct a chain homotopy $\big(h_n : C_n(M) \to C_{n+1}(M)\big)_{n \geqslant 0}$ from $(i_0 r)_\bullet$ to $\mathrm{id}_{C_\bullet(M)}$, we proceed by induction on n.

$$\text{Define} \quad \begin{aligned} h_0\big((v,0)\big) &= 0 \\ h_0\big((\widehat{\sigma}, \tfrac{1}{2})\big) &= \big\langle (a(\widehat{\sigma}), 0), (\widehat{\sigma}, \tfrac{1}{2}) \big\rangle \\ h_0\big((v,1)\big) &= \big\langle (v,0), (v, \tfrac{1}{2}) \big\rangle + \big\langle (v, \tfrac{1}{2}), (v, 1) \big\rangle \end{aligned}$$

For $n > 0$, assume h_{n-1} is defined and satisfies $dh_{n-1} + h_{n-2}d = \mathrm{id}_{C_{n-1}(M)} - (i_0 r)_{n-1}$, and that for each simplex π of M we have $h_{n-1}(\pi) \in C_n(M_\sigma)$, where σ is the appropriate simplex of K.

Now let π be an n-simplex of M with associated simplex $\sigma \in K$, and consider the chain $c = \pi - (i_0 r)_n(\pi) - h_{n-1} d(\pi) \in C_n(M_\sigma)$.

$$d(c) = d(\pi) - (i_0 r)_{n-1} d(\pi) - dh_{n-1} d(\pi) = h_{n-2} d^2(\pi) = 0,$$

so $c \in Z_n(M_\sigma) = B_n(M_\sigma)$, by **5.10**.

Hence we can choose $x \in C_{n+1}(M_\sigma)$ with $d(x) = c$, and define $h_n(\pi) = x$.

Thus $(i_0)_* = (i_1)_* : H_n(K) \to H_n(M)$, since they have the same two-sided inverse r_*.

Now, $H : |M| \to |L|$ has a simplicial approximation $\widetilde{H} : M^{(r)} \to L$ for some $r \geqslant 0$. We may choose $\widetilde{H}$ so that its restrictions to $|K| \times \{0\}$ and $|K| \times \{1\}$ coincide with fa^r and ga^r respectively, where $a^r : K^{(r)} \to K$ is a simplicial approximation to $\mathrm{id}_{|K|}$.

Thus $\begin{array}{ccc} K^{(r)} & \xrightarrow{a^r} & K \\ i_0^{(r)}\downarrow & & \downarrow f \\ M^{(r)} & \xrightarrow{\widetilde{H}} & L \end{array}$ commutes. $(*)$ (So does a similar diagram for g.)

This induces a commutative diagram of homology groups.

But $\begin{array}{ccc} K^{(r)} & \xrightarrow{a^r} & K \\ i_0^{(r)}\downarrow & & \downarrow i_0 \\ M^{(r)} & \xrightarrow{a^r} & M \end{array}$ also commutes. $(**)$

Thus $H_n(f)$ may be written as the composites

$$\begin{array}{lccccccccl} & H_n(K) & \xrightarrow{(s^r)_*} & H_n(K^{(r)}) & \xrightarrow{(a^r)_*} & H_n(K) & \xrightarrow{f_*} & H_n(L) & \\ \text{i.e.,} & H_n(K) & \xrightarrow{(s^r)_*} & H_n(K^{(r)}) & \xrightarrow{(i_0^{(r)})_*} & H_n(M^{(r)}) & \xrightarrow{\widetilde{H}_*} & H_n(L), & \text{by } (*) \\ \text{i.e.,} & H_n(K) & \xrightarrow{(i_0)_*} & H_n(M) & \xrightarrow{(s^r)_*} & H_n(M^{(r)}) & \xrightarrow{\widetilde{H}_*} & H_n(L), & \text{by } (**) \end{array}$$

And $(i_0)_* = (i_1)_*$, so $H_n(f) = H_n(g)$, and so $f_* = g_*$. □

5.14 Theorem. For $n \geqslant 0$, we have a functor H_n from {triangulable spaces} $\to$ {abelian groups}. Moreover, $f \simeq g$ implies $H_n(f) = H_n(g)$.

Proof. Given a triangulable space X, we define $H_n(X)$ to be $H_n(K)$ for some simplicial complex K with $|K| \cong X$ (i.e. homeomorphic).

Given $f : |K| \to |L|$, we define $H_n(f)$ to be $H_n(K) \xrightarrow{(s^r)_*} H_n(K^{(r)}) \xrightarrow{\widetilde{f}_*} H_n(L)$ for some simplicial approximation $\widetilde{f} : K^{(r)} \to L$ to f. By **5.13** and the remarks before it, this is independent of the choice of $\widetilde{f}$.

If $|K| \xrightarrow{f} |L| \xrightarrow{g} |M|$ are two simplicial maps, we can choose simplicial approximations $\widetilde{g} : L^{(r_1)} \to M$ to g, and $\widetilde{f} : K^{(r_2)} \to L^{(r_1)}$ to f.

Then $a^{r_1}\widetilde{f} : K^{(r_1)} \to L$ is also a simplicial approximation to f, so that composite $g_* f_*$ may be written as

$$H_n(K) \xrightarrow{(s^{r_2})_*} H_n(K^{(r_2)}) \xrightarrow{\widetilde{f}_*} H_n(L^{(r_1)}) \xrightarrow{(a^r)_*} H_n(L) \xrightarrow{(s^{r_1})_*} H_n(L^{(r_1)}) \xrightarrow{g_*} H_n(M)$$

I.e., as

$$H_n(K) \xrightarrow{(s^{r_2})_*} H_n(K^{(r_2)}) \xrightarrow{\widetilde{f}_*} H_n(L^{(r_1)}) \xrightarrow{g_*} H_n(M)$$

But $\widetilde{g}\widetilde{f}$ is a simplicial approximation to gf, so this is $(gf)_*$.

Hence also $H_n(X)$ is well-defined: for if K and L are two triangulations of X, we have $|K| \cong X \cong |L|$, so by functoriality $H_n(K) \cong H_n(L)$.

Finally, homotopy invariance follows from **5.13** and the remarks before it. □

Chapter 6 : Applications of Homology Groups

Recall that in **5.11** we calculated the homology groups of S^n for any $n \geqslant 1$:

$$H_r(S^n) = \begin{cases} \mathbb{Z} & \text{if } r = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

For completeness, we note that $S^0 = \{\pm 1\}$ has

$$H_r(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

6.1 Proposition.

(i) If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism then $m = n$.

(ii) There is no continuous retraction $B^n \to S^{n-1}$.

Proof.

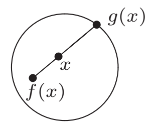
(i) We can restrict f to a homeomorphism $\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$.
But $\mathbb{R}^n \setminus \{x\} \simeq S^{n-1}$, so f induces a homotopy equivalence $S^{m-1} \simeq S^{n-1}$ and hence isomorphisms $H_r(S^{m-1}) \cong H_r(S^{n-1})$ for all r. And this forces $m = n$.

(ii) Let $i : S^{n-1} \hookrightarrow B^n$ be the inclusion map. Then $i_* : H_{n-1}(S^{n-1}) \to H_{n-1}(B^n)$ isn't injective. So there is no $r : B^n \to S^{n-1}$ with $ri = \mathrm{id}_{S^{n-1}}$. $\square$

6.2 Brouwer's fixed point theorem. Every continuous $f : B^n \to B^n$ has a fixed point.

Proof. Suppose f is continuous and has no fixed point.

Define $g : B^n \to S^{n-1}$ as follows: $g(x)$ is the point where the line from $f(x)$ to x, extended, meets the sphere.



It's clear that the continuity of f implies continuity of g, and if $\|x\| = 1$ then $g(x) = x$, so g is a retraction. $\lightning$ $\square$

By similar arguments, we can prove the **Borsuk-Ulam theorem**: for any continuous $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$. (See example sheet 4, question 6.)

To calculate homology groups of more complicated spaces, we need a homology analogue of Seifert–Van Kampen, allowing us to calculate $H_*(X)$ from $H_*(Y)$, $H_*(Z)$ and $H_*(Y \cap Z)$, if $X = Y \cup Z$.

6.3 Definition. Let $\ldots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \ldots$ be a (finite or infinite) sequence of abelian groups and homomorphisms.

We say the sequence is **exact at** A_n if $\ker f_n = \operatorname{im} f_{n+1}$. We say the sequence is **exact** if it is exact at every A_n (other than the end-points, if they exist).

Example. The sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact iff f is injective, g is surjective, and $C \cong B/f(A)$.

6.4 Theorem (Mayer-Vietoris). Suppose X is the union of two subspaces Y and Z, and suppose X has a triangulation K such that $Y = |L|$ and $Z = |M|$ for suitable subcomplexes L and M of K.

Denote the inclusion maps by
$$\begin{array}{ccc} Y\cap Z & \xrightarrow{i} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{l} & X \end{array}$$

Then for each $n \geqslant 1$, there exists a homomorphism $\partial_* : H_n(X) \to H_{n-1}(Y \cap Z)$ making the following sequence exact.

$$\begin{tikzcd} \ldots \arrow[r] & H_n(Y\cap Z) \arrow[r, "(i_*,j_*)"] & H_n(Y)\oplus H_n(Z) \arrow[r, "k_*-l_*"] & H_n(X) \arrow[dll, "\partial_*"] \\ & H_{n-1}(Y\cap Z) \arrow[r, "(i_*,j_*)"] & H_{n-1}(Y)\oplus H_{n-1}(Z) \arrow[r, "k_*-l_*"] & \ldots \\ & \ldots \arrow[r, "(i_*,j_*)"] & H_0(Y)\oplus H_0(Z) \arrow[r, "k_*-l_*"] & H_0(X) \arrow[r] & 0 \end{tikzcd}$$

Proof. Observe that for each n we have an exact sequence

$$0 \longrightarrow C_n(L\cap M) \xrightarrow{(i_n,j_n)} C_n(L)\oplus C_n(M) \xrightarrow{k_n-l_n} C_n(K) \longrightarrow 0$$

Now, (i_n, j_n) is injective since both i_n and j_n are, and $k_n - l_n$ is surjective since $K = L \cup M$ and so any chain on K is the sum of a chain on L on one on M, and

$$\begin{aligned} \ker(k_n - l_n) &= \{(c,d) : c \in C_n(L),\ d \in C_n(M), \text{ and } c = d \text{ in } C_n(K)\} \\ &= \{(c,c) : c \in C_n(L\cap M)\} = \operatorname{im}(i_n, j_n) \end{aligned}$$

Thus we're reduced to proving the following algebraic statement.

Given chain complexes $C_\bullet$, $D_\bullet$, $E_\bullet$, and chain maps $C_\bullet \xrightarrow{f_\bullet} D_\bullet \xrightarrow{g_\bullet} E_\bullet$ such that

$$0 \longrightarrow C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \longrightarrow 0$$

is exact for all n, we get a long exact sequence

$$\ldots \longrightarrow H_n(C_\bullet) \xrightarrow{f_*} H_n(D_\bullet) \xrightarrow{g_*} H_n(E_\bullet) \xrightarrow{\partial_*} H_{n-1}(C_\bullet) \xrightarrow{f_*} H_{n-1}(D_\bullet) \longrightarrow \ldots$$

We do this in several stages.

Step 1. Constructing ∂_*.

Let $x \in H_n(E_\bullet)$ and let $y \in Z_n(E_\bullet)$ be a cycle representing x. Choose $z \in D_n$ with $g_n(z) = y$.

Now $g_{n-1}d(z) = dg_n(z) = d(y) = 0$, so $d(z) \in \ker g_{n-1} = \operatorname{im} f_{n-1}$. So there's a unique $w \in C_{n-1}$ with $f_{n-1}(w) = d(z)$.

Now $f_{n-2}d(w) = df_{n-1}(w) = d^2(z) = 0$, and f_{n-2} is injective, so $d(w) = 0$. So $w \in Z_{n-1}(C_\bullet)$.

We define $\partial_*(x)$ to be the homology class represented by w.

Step 2. ∂_* is well-defined.

Suppose we made different choices y', z', w' in Step 1. Then $y - y' \in B_n(E_\bullet)$, so we can find $t \in E_{n+1}$ with $d(t) = y - y'$. Then we can find $u \in D_{n+1}$ with $g_{n+1}(u) = t$.

Now consider $z - z' - d(u) \in D_n$. We have $g_n(z - z' - d(u)) = y - y' - dg_{n+1}(u) = y - y' - d(t) = 0$, so there exists $v \in C_n$ with $f_n(v) = z - z' - d(u)$.

Now $f_{n-1}d(v) = df_n(v) = d(z) - d(z') = f_{n-1}(w) - f_{n-1}(w')$, and f_{n-1} is injective, so $d(v) = w - w'$. Hence w and w' represent the same homology class.

$$
\begin{array}{ccccccccc}
 & & & & u & \xrightarrow{g_{n+1}} & t & & \\
 & & & & \searrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
 & & v & \longrightarrow & z-z'-d(u) & & y-y' & & \\
0 & \longrightarrow & C_n & \xrightarrow{f_n} & D_n & \xrightarrow{g_n} & E_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} & \xrightarrow{g_{n-1}} & E_{n-1} & \longrightarrow & 0 \\
 & & w-w' & \longrightarrow & d(z-z') & & & &
\end{array}
$$

Step 3. ∂_* is a homomorphism.

If y_1, z_1, w_1 are suitable choices for $x_1 \in H_n(E_\bullet)$ and y_2, z_2, w_2 are suitable choices for $x_2 \in H_n(E_\bullet)$, then $y_1 + y_2$, $z_1 + z_2$, $w_1 + w_2$ are suitable choices for $x_1 + x_2$. So $\partial_*(x_1 + x_2) = \partial_*(x_1) + \partial_*(x_2)$.

Step 4. The sequence is exact at $H_n(D_\bullet)$.

We know $g_* f_* = 0$ since $g_\bullet f_\bullet = 0$ and the passage from $f_\bullet$ to f_* is functorial. So we need only show $\ker g_* \subset \operatorname{im} f_*$, so let $z \in Z_n(D_\bullet)$ be such that $g_*([z]) = 0$.

Then $g_n(z)$ is a boundary, say $g_n(z) = d(t)$ for some $t \in E_{n+1}$, and g_{n+1} is surjective so there exists $u \in D_{n+1}$ with $g_{n+1}(u) = t$. We have $g_n(z - d(u)) = g_n(z) - dg_{n+1}(u) = 0$.

So $z - d(u) \in \ker g_n = \operatorname{im} f_n$, so we have a unique $v \in C_n$ with $f_n(v) = z - d(u)$. Then $f_{n-1}d(v) = df_n(v) = d(z - d(u)) = 0$ since z is a cycle. And f_{n-1} is injective, so v is a cycle and $f_*([v]) = [z - d(u)] = [z]$.

Step 5. The sequence is exact at $H_n(E_\bullet)$.

Suppose $x = [y]$ is in the image of $g_* : H_n(D_\bullet) \to H_n(E_\bullet)$. Then in the construction of $\partial_*(x)$ we can choose $z \in D_n$ to be a cycle with $g_n(z) = y$. This gives $d(z) = 0$, so $w = 0$, and so $\partial_*(x) = [w] = 0$.

Conversely, suppose $\partial_*(x) = 0$. Then the element $w \in C_{n-1}$ which we can construct must be a boundary, say $w = d(v)$ for some $v \in C_n$.

Now $d(z - f_n(v)) = d(z) - f_{n-1}d(v) = 0$, so $z - f_n(v) \in Z_n(D_\bullet)$, and $g_n(z - f_n(v)) = g_n(z)$ since $g_n f_n = 0$. So $[z - f_n(v)]$ is a homology class mapped by g_* to x.

Step 6. The sequence is exact at $H_{n-1}(C_\bullet)$.

If $[w] = \partial_*(x)$ then $f_*([w]) = [f_{n-1}(w)] = [d(z)] = 0$, so $\operatorname{im} \partial_* \subset \ker f_*$.

Conversely, suppose $[w] \in H_{n-1}(C_\bullet)$ and $f_*([w]) = 0$. Then $f_{n-1}(w) = d(z)$ for some $z \in D_n$, and $y = g_n(z)$ is a cycle, since $d(y) = g_{n-1}d(z) = g_{n-1}f_{n-1}(w) = 0$.

So the homology class $x = [y]$ satisfies $\partial_*(x) = [w]$. □

6.5 Remark. A polyhedron $|K|$ is always locally path-connected, so it's globally path-connected if and only if it's connected, and if and only if for any two vertices v_0, v_1 of K there is a 1-chain c with $d(c) = v_1 - v_0$.

Hence, for a connected polyhedron K, we have $H_0(K) \cong \mathbb{Z}$, and the quotient map $Z_0(K) = C_0(K) \to H_0(K)$ is given by $\sum_{i=1}^m n_i v_i \mapsto \sum_{i=1}^m n_i$.

More generally, for an arbitrary polyhedron X, we have $H_0(X) \cong \mathbb{Z}^r$ where r is the number of connected components of X.

So in the Mayer-Vietoris sequence for $X = Y \cup Z$, if $Y \cap Z$ is connected then $H_0(Y \cap Z) \to H_0(Y) \oplus H_0(Z)$ is injective, and so $\partial_* : H_1(X) \to H_0(Y \cap Z)$ is the zero homomorphism.

So we could terminate the sequence at $\ldots \longrightarrow H_1(Y) \oplus H_1(Z) \longrightarrow H_1(X) \longrightarrow 0$.

6.6 Examples.

(a) Suppose $X = Y \vee Z$ is a wedge union of two polyhedra. We may assume that Y and Z have been triangulated so that the identified point is a vertex of each of them, and then the union gives us a triangulation of X.

So we can use Mayer-Vietoris to calculate $H_*(X)$, and in all dimensions > 0 we have exact sequences of the form

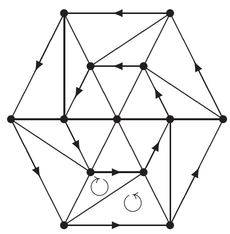
$$0 \longrightarrow H_n(Y) \oplus H_n(Z) \longrightarrow H_n(X) \longrightarrow 0$$

(For $n = 1$, this uses **6.5**.) So $H_n(X) \cong H_n(Y) \oplus H_n(Z)$ for all $n > 0$.

And if Y and Z are connected then so is X, so $H_0(X) \cong \mathbb{Z}$.

(b) Consider the Lens space, $L_n = B^2/\sim$, where $\sim$ identifies points z_1, z_2 of S^1 if $z_1^n = z_2^n$. In **3.5**(b) we calculated $\Pi_1(L_n)$ using Seifert–Van Kampen.

We triangulate L_n, as shown here for $n = 6$.



Let M be the central region, so $|M| \cong B^2$. Let N be the closure of $L_n \setminus |M|$, so $|N| \cong S^1$ and $|M \cap N| \cong S^1$.

So $H_1(M \cap N) \cong \mathbb{Z}$, generated by a cycle which is the sum of all its 1-simplices, oriented as shown.

In $H_1(N)$ this cycle represents the same homology class as the cycle going around the outer boundary, as their difference is the boundary of an appropriate 2-cycle. But this is n times the generator of $H_1(N)$, so in Mayer-Vietoris, we get

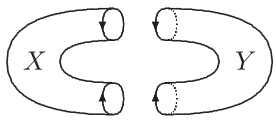
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(L_n) & \longrightarrow & H_1(M \cap N) & \xrightarrow{\times n} & H_1(N) & \longrightarrow & H_1(L_n) & \longrightarrow & 0 \\ & & & & \| & & \| & & & & \\ & & & & \mathbb{Z} & & \mathbb{Z} & & & & \end{array}$$

So we get $H_2(L_n) = 0$ and $H_1(L_n) \cong \mathbb{Z}/n\mathbb{Z}$.

(We also have $H_0(L_n) = \mathbb{Z}$, and $H_r(L_n) = 0$ for all $r \geqslant 3$.)

In particular, since $\mathbb{RP}^2 \cong L_2$, we have $H_r(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$

(c) Consider the torus T as $X \cup Y$, where X and Y are homeomorphic to $S^1 \times I$, and $X \cap Y$ is the union of two disjoint copies of S^1.



Then $H_0(X \cap Y) \cong \mathbb{Z} \oplus \mathbb{Z} \cong H_0(X) \oplus H_0(Y)$.

But the homomorphism $(i_*, j_*) : H_0(X \cap Y) \to H_0(X) \oplus H_0(Y)$ sends (p, q) to $(p+q, p+q)$. So $\ker(i_*, j_*) \cong \mathbb{Z}$, generated by $(1, -1)$.

And $H_0(T) \cong H_0(X) \oplus H_0(Y)/\mathrm{im}\,(i_*, j_*) \cong \mathbb{Z}$.

We also have $H_1(X \cap Y) \cong \mathbb{Z} \oplus \mathbb{Z} \cong H_1(X) \oplus H_1(Y)$. The generators of $H_1(X \cap Y)$ are the two cycles shown, and these have the same image under i_* since their difference is a boundary in $C_0(X)$. Similarly, their images under j_* are equal.

So $(i_*, j_*) : H_1(X \cap Y) \to H_1(X) \oplus H_1(Y)$ is again $(p, q) \mapsto (p+q, p+q)$.

Now we have:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H_2(T) & \longrightarrow & H_1(X \cap Y) & \longrightarrow & H_1(X) \oplus H_1(Y) \\
 & & \| & & \| & & \| \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\
 & & & & (p, q) & \longmapsto & (p+q, p+q)
\end{array}
$$

$$
\begin{array}{ccccccccc}
\longrightarrow H_1(T) & \longrightarrow & H_0(X \cap Y) & \longrightarrow & H_0(X) \oplus H_0(Y) & \longrightarrow & H_0(T) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & (p, q) & \longmapsto & (p+q, p+q) & & & &
\end{array}
$$

And so we have an exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{u} H_1(T) \xrightarrow{v} \mathbb{Z} \longrightarrow 0$.

Choose $x \in H_1(T)$ with $v(x) = 1$. Then $\langle v \rangle$ is a subgroup of $H_1(T)$ isomorphic to $\mathbb{Z}$, and $H_1(T) \cong \langle u(1) \rangle \oplus \langle v \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.

6.7 Remark. It can be shown that for any connected polyhedron X, $H_1(X)$ is isomorphic to the quotient of $\Pi_1(X)$ by its commutator subgroup, i.e. the largest abelian quotient of $\Pi_1(X)$. (See example sheet 3, question 11.)

In particular, for two such spaces X and Y, if $\Pi_1(X) \cong \Pi_1(Y)$ then $H_1(X) \cong H_1(Y)$.

Classification of triangulable 2-manifolds

(For the detailed proofs, see C. R. F. Maunder, *Algebraic Topology*.)

6.8 Definition. By an n**-manifold**, we mean a Hausdorff topological space X such that every point of X has a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$.

(The Hausdorff condition excludes things like $\mathbb{R} \times \{0, 1\}/\sim$, where $\sim$ identifies $(x, 0)$ with $(x, 1)$ for all $x \neq 0$.)

We consider triangulable manifolds, which are necessarily metrizable and compact. (E.g., S^n is a triangulable n-manifold, but $\mathbb{R}^n$ is not.)

6.9 Lemma. Suppose K is a simplicial complex. The following are equivalent.

(i) $|K|$ is a manifold

(ii) For every $x \in |K|$, $st_K(x)$ is homeomorphic to an open ball in $\mathbb{R}^n$

(iii) For every $x \in |K|$, $lk_K(x)$ is homeomorphic to S^{n-1}.

6.10 Corollary. If $|K|$ is an n-manifold, then

(i) Every simplex of K is a face of an n-simplex

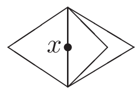
(ii) Every $(n-1)$-simplex of K is a face of exactly two n-simplices.

Proof.

(i) It is enough to show that every maximal simplex of K has dimension n. But if σ is a maximal simplex, say of dimension m, then for any $x \in \sigma^\circ$, we have $st_K(x) = \sigma^\circ$ and $lk_K(x) = \partial\sigma \cong S^{n-1}$, So $m = n$.

(ii) Let τ be an $(n-1)$-simplex of K, and consider $lk_K(x)$ for any $x \in \tau^\circ$.

If τ is a face of r n-simplices $\sigma_1, \ldots, \sigma_r$, then $lk_K(x)$ is the union of r copies of B^{n-1} with their boundaries (each S^{n-2}) identified.



E.g., $n = 2$, $r = 3$

By shrinking one of the B^{n-1} to a point, we see that $lk_K(x) \cong \bigvee_{i=1}^{r-1} S^{n-1}$, and by considering H_{n-1} we see that $r = 2$. □

6.11 Corollary. For a simplicial complex K, $|K|$ is a 2-manifold (i.e., a **surface**) iff

(i) All simplices of K have dimension $\leqslant 2$

(ii) Every 1-simplex of K is a face or two 2-simplices

(iii) For every vertex v of K, $lk_K(v)$ is connected.

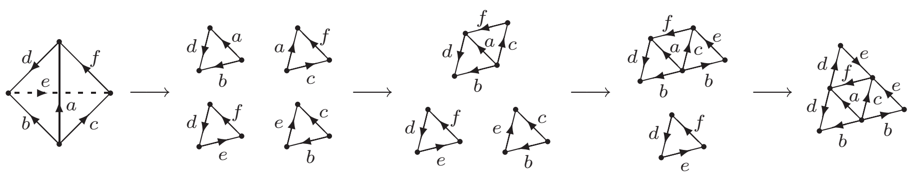
Proof. (i) and (ii) imply that the link of any point other than a vertex in $|K|$ is homeomorphic to S^1. But (i) and (ii) also imply that the link of any vertex v of K is a 1-manifold, since each vertex in it is a face of exactly two 1-simplices in it. But a triangulable 1-manifold is a disjoint union of copies of S^1, so connectedness in (iii) implies that it's a single copy of S^1.

So we've shown that (i), (ii), (iii) imply condition (iii) of **6.9**. The converse is easy. □

6.12 Lemma. Every connected triangulable 2-manifold is homeomorphic to a space obtained from a $2n$-gon by identifying its edges in pairs.

Proof. Let K be a triangulation of our manifold. First mark each 1-simplex of K with a unique label $a, b, c, \ldots$, and an orientation.

Next, pull the 2-simplices apart, and then reassemble them in the plane as follows. At each stage, look for a label that appears once on the boundary of what we've got so far, find the other simplex with that label and paste it on.



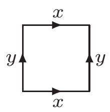
We need to show that this process stops, i.e. that when each label on the boundary of our polygon appears twice, we've used up all of the 2-simplices of K.

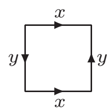
Suppose not. Let L denote the subcomplex of K formed by the simplices we've used so far, together with their faces. Since every 1-simplex of L is a face of two 2-simplices of L, a 2-simplex of $K \setminus L$ must have all of its 1-faces in $K \setminus L$.

If its vertices are not in L, we can join one of them to a vertex of L by a chain of 1-simplices of K. Hence there's a 1-simplex τ of $K \setminus L$ which has at least one vertex v in L. The other vertex, w say, of τ is in $lk_K(v) \setminus lk_L(v)$.

By **6.11**(iii) we can join it to a vertex in $lk_L(v)$ by a chain of 1-simplices in $lk_K(v)$. Again, there will be a 1-simplex, π say, in this chain which doesn't lie in $lk_L(v)$ but has a vertex, u say, in $lk_L(v)$. Now the 1-simplex $\langle v, u\rangle$ is a face of the 2-simplex $\langle v, \pi\rangle$ as well as of two 2-simplices in L. ⨳ □.

We can represent a polygon with identification as in **6.12** by a word of $2n$ 'letters', each of which is an x or an x^{-1}, in which each letter appears twice.

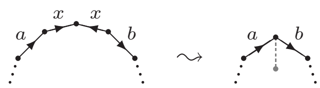
E.g., the torus 

corresponds to $xyx^{-1}y^{-1}$ and the Klein bottle 

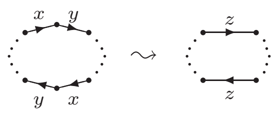
to $xyx^{-1}y$.

Many different words can represent the same 2-manifold: we can permute the letters cyclically, or we can replace a word by its inverse.

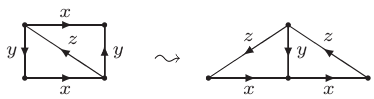
Also, if we have $\ldots xx^{-1}\ldots$ in our word, we can cancel the x and x^{-1}, provided this leaves at least four letters.



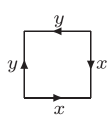
Similarly, if we have $\ldots xy\ldots xy\ldots$ or $\ldots xy\ldots y^{-1}x^{-1}\ldots$ in our word, then we can replace xy by a new letter z, provided we still have at least four letters.

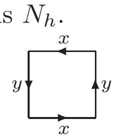


Also, we can slice a polygon between a pair of vertices and re-glue it along a pair of identified edges.



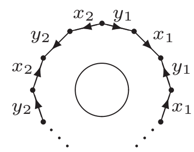
6.13 Proposition. Using moves of the above kinds, we can reduce any word to one in the following list.

- If all letters in our word are balanced, i.e. occur in pairs (x, x^{-1}), then we can reduce to one of
  - $x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}\ldots x_gy_gx_g^{-1}y_g^{-1}$, for some $g \geqslant 1$. Call this M_g.
  - $xx^{-1}yy^{-1}$ if $g = 0$. Note that this is S^2 : 

- If there's an unbalanced letter then the word reduces to one of
  - $x_1x_1x_2x_2\ldots x_hx_h$, for some $h \geqslant 2$. Call this N_h.
  - $xyxy$, if $h = 1$. This corresponds to $\mathbb{RP}^2$: 

Proof. Omitted. □

We can show that these words correspond to distinct manifolds by calculating their homology groups.



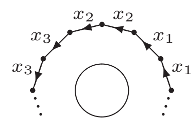
For M_g, we express it as the union of a disc B^2 and the annulus with identifications Y. Note that all vertices on the boundary are identified and hence Y is homotopy equivalent to a wedge of $2g$ circles.

$B^2 \cap Y \simeq S^1$, and the cycle representing the generator of $H_1(B^2 \cap Y)$ is homologous to the cycle going around the outer boundary of Y, but this cycle is 0 in $H_1(Y)$. So we have

$$\begin{tikzcd}
0 \arrow[r] & H_2(M_g) \arrow[r] & H_1(B^2 \cap Y) \arrow[r] \arrow[d, equal] & H_1(Y) \arrow[r] \arrow[d, equal] & H_1(M_g) \arrow[r] & 0 \\
& & \mathbb{Z} & \mathbb{Z}^{2g} & &
\end{tikzcd}$$

and the middle map is zero. So $H_2(M_g) \cong \mathbb{Z}$, $H_1(M_g) \cong \mathbb{Z}^{2g}$, and $H_0(M_g) \cong \mathbb{Z}$.

(Note that this also holds for $g = 0$.)



Similarly, we calculate the homology of N_h. Here again, all vertices around the boundary are identified, so $Y \simeq \bigvee_{i=1}^h S^1$.

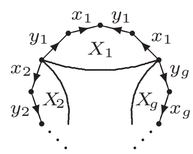
$B^2 \cap Y \simeq S^1$, but the inclusion $B^2 \cap Y \hookrightarrow Y$ sends a generator of $H_1(B^2 \cap Y)$ to $2(x_1 + \ldots + x_h)$, where $x_1, \ldots, x_h$ are the generators of $H_1(Y) \cong \mathbb{Z}^h$. So we have

$$\begin{tikzcd}
0 \arrow[r] & H_2(N_h) \arrow[r] & H_1(B^2 \cap Y) \arrow[r] \arrow[d, equal] & H_1(Y) \arrow[r] \arrow[d, equal] & H_1(N_h) \arrow[r] & 0 \\
& & \mathbb{Z} & \mathbb{Z}^{h} & &
\end{tikzcd}$$

The middle map is injective, so $H_2(N_h) = 0$, and $H_1(N_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

(Again, this works for $h = 0$.)

What does M_g look like geometrically? Cut it up like this:



After identification of boundary vertices, Y is homeomorphic to $S^2 \setminus (g$ disjoint open discs).

Each X_i is homeomorphic to a torus minus an open disc, or equivalently to a 'handle':

So M_g is homeomorphic to a sphere with g handles attached.

Similarly, N_h is homeomorphic to a sphere with h cross-caps attached. (A **cross-cap** is a Möbius band whose boundary circle has been identified with the boundary of a hole in the sphere.)

6.14 Remark. For any connected (triangulable) n-manifold M, we have either $H_n(M) \cong \mathbb{Z}$ or $H_n(M) = 0$. (See example sheet 2, question 9.)

In the former case, we say M is **orientable**, and by an **orientation** of M we mean a choice of generator for $H_n(M)$ (equivalently, a compatible choice of orientation for all of the n-simplices of M).

For M_g, we can orient all 2-simplices anticlockwise as seen from the outside – this gives one possible orientation.

6.15 Definition. Let K be a simplicial complex. By a **rational n-chain** on K we mean a formal sum $\sum_{i=1}^m q_i\sigma_i$, where the σ_i are oriented n-simplices of K and $q_i \in \mathbb{Q}$.

The set of all rational n-chains forms a $\mathbb{Q}$-vector space $C_n(K,\mathbb{Q})$, where we write $C_n(K,\mathbb{Z})$ for the group of integer n-chains as previously considered.

We can define $d : C_n(K,\mathbb{Q}) \to C_{n-1}(K,\mathbb{Q})$ as before, and $d^2 = 0$ as before, so we can define $Z_n(K,\mathbb{Q})$, $B_n(K,\mathbb{Q})$ and $H_n(K,\mathbb{Q})$ as before. All of the results of chapter 5 remain true for rational homology, as does the Mayer-Vietoris theorem.

6.16 Lemma. Suppose $H_n(X,\mathbb{Z}) \cong \mathbb{Z}^r \oplus F$, where F is a finite group. Then $H_n(X,\mathbb{Q}) \cong \mathbb{Q}^r$.

Proof. Suppose K is a triangulation of X, and let $c \in Z_n(K,\mathbb{Z})$. If $[c]$ has finite order in $H_n(K,\mathbb{Z})$ then $mc \in B_n(K,\mathbb{Z})$ for some $m > 0$, so $c \in B_n(K,\mathbb{Q})$.

However, if $c_1,\ldots,c_r$ are cycles representing a set of generators for $\mathbb{Z}^r \subset H_n(K,\mathbb{Z})$, they remain linearly independent over $\mathbb{Q}$, and $[c_1],\ldots,[c_r]$ form a basis for $H_n(K,\mathbb{Q})$.$\square$

6.17 Definition. Let X be a polyhedron and $f : X \to X$ a continuous map.

The **Lefschetz number**, $L(f)$, of f is

$$\sum_{i\geqslant 0}(-1)^i \operatorname{trace}\big(f_* : H_i(X,\mathbb{Q}) \to H_i(X,\mathbb{Q})\big).$$

The **Euler-Poincaré characteristic**, $\chi(X)$, of X is

$$L(\mathrm{id}_X) = \sum_{i\geqslant 0}(-1)^i \dim_{\mathbb{Q}} H_i(X,\mathbb{Q}).$$

For example, $\chi(M_g) = 2-2g$ and $\chi(N_h) = 2-h$.

6.18 Lemma. Let V be a vector space, W a subspace of V, and f a linear map $V \to V$ such that $f(W) \subset W$. Write $f_1 : W \to W$ for $f|_W$ and $f_2 : V/W \to V/W$ for the induced map. Then $\mathrm{tr} f = \mathrm{tr} f_1 + \mathrm{tr} f_2$.

Proof. Let $e_1,\ldots,e_r$ be a basis for W, and extend to a basis $e_1,\ldots,e_n$ for V. The matrix of f with respect to this basis has the form

$$\left(\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right) \quad \text{where} \quad \begin{array}{l} B = \text{matrix of } f_1 \text{ w.r.t. } \{e_1,\ldots,e_r\} \\ D = \text{matrix of } f_2 \text{ w.r.t. } \{e_{r+1}+W,\ldots,e_n+W\} \end{array}$$

So $\mathrm{tr} f = \mathrm{tr} B + \mathrm{tr} D = \mathrm{tr} f_1 + \mathrm{tr} f_2$. $\square$

6.19 Corollary. Let $f : K \to K$ be a simplicial map. Then

$$\sum_{i\geqslant 0}(-1)^i \operatorname{tr}\big(f_* : H_i(X,\mathbb{Q}) \to H_i(X,\mathbb{Q})\big) = \sum_{i\geqslant 0}(-1)^i \operatorname{tr}\big(f_i : C_i(X,\mathbb{Q}) \to C_i(X,\mathbb{Q})\big).$$

Proof. Let: $f_{i,1}$ be the restriction of f_i to $B_i(K,\mathbb{Q})$,
$f_{i,2}$ be the induced map of f_i on $Z_i(K,\mathbb{Q})/B_i(K,\mathbb{Q})$,
$f_{i,3}$ be the induced map of f_i on $C_i(K,\mathbb{Q})/Z_i(K,\mathbb{Q})$.

Then $\mathrm{tr} f_i = \mathrm{tr} f_{i,1} + \mathrm{tr} f_{i,2} + \mathrm{tr} f_{i,3}$.

But we have an isomorphism $C_i/Z_i \to B_{i-1}$ making the diagram

$$\begin{array}{ccc} C_i/Z_i & \longrightarrow & B_{i-1} \\ {\scriptstyle f_{i,3}} \downarrow & & \downarrow {\scriptstyle f_{i-1,1}} \\ C_i/Z_i & \longrightarrow & B_{i-1} \end{array}$$

commute. So $\mathrm{tr} f_{i,3} = \mathrm{tr} f_{i-1,1}$.

So, in the sum $\sum (-1)^i \mathrm{tr} f_i$, the terms involving $\mathrm{tr} f_{i,1}$ and $\mathrm{tr} f_{i,3}$ all cancel. □

6.20 Corollary. For any simplicial complex K, we have $\chi(|K|) = \sum_{i \geqslant 0} (-1)^i n_i$, where n_i is the number of i-simplices in K.

In particular, this alternating sum is independent of the choice of triangulation of $|K|$.

6.21 Remark. Let X, Y be polyhedra and suppose that $p : X \to Y$ is an n-to-1 covering projection. Then $\chi(X) = n\chi(Y)$.

Proof. We can choose our triangulation of Y so that each simplex is evenly covered by p, and so its inverse image is the disjoint union of n simplices of the same dimension. Hence we have a triangulation of X with n times as many i-simplices as in that of Y, for each $i \geqslant 0$. □

6.22 Theorem (Lefschetz fixed-point theorem). Let X be a polyhedron and $f : X \to X$ a continuous map such that $L(f) \neq 0$. Then f has a fixed point.

Proof. We prove the contrapositive. Suppose $f : X \to X$ has no fixed points. Consider $\delta = \inf \big\{ d(x, f(x)) : x \in X \big\}$, where d is the Euclidean distance on X. By compactness of K, this infimum is attained, so $\delta > 0$.

Let K be a triangulation of X with $\mathrm{mesh}(K) \leqslant \frac{\delta}{2}$. (Possible by **4.17**.) Choose a simplicial approximation $g : K^{(r)} \to K$ to f, for some $r \geqslant 0$.

For each vertex v of $K^{(r)}$, we have $f(v) \in f\big(st_{K^{(r)}}(v)\big) \subset st_K(g(v))$ and hence $d(f(v), g(v)) < \frac{\delta}{2}$. But $d(v, f(v)) \geqslant \delta$, so $d(v, g(v)) > \frac{\delta}{2}$. In particular, if v lies in a simplex σ of K, then $g(v) \notin \sigma$.

The effect of f on homology groups of K is induced by the chain map

$$C_\bullet(K, \mathbb{Q}) \xrightarrow{s_\bullet^r} C_\bullet(K^{(r)}, \mathbb{Q}) \xrightarrow{g_\bullet} C_\bullet(K, \mathbb{Q})$$

Hence by the argument of **6.19** we can compute $L(f)$ as

$$\sum_{i \geqslant 0} (-1)^i \mathrm{tr}\big(g_i s_i^r : C_i(K, \mathbb{Q}) \to C_i(K, \mathbb{Q})\big)$$

Given an i-simplex σ of K, s_i^r sends it to a chain which is the sum of all the i-simplices of $K^{(r)}$ contained in σ, with suitable orientations.

But g_i sends all of these either to 0 or to simplices of K disjoint from σ. Hence the matrix of $g_i s_i^r$ (w.r.t. the obvious basis of $C_i(K, \mathbb{Q})$) has all diagonal entries 0.

Hence $\mathrm{tr}(g_i s_i^r) = 0$, and so $L(f) = 0$. □

6.23 Examples.

(a) Lefschetz $\Rightarrow$ Brouwer. More generally, if X is a contractible polyhedron, then *any* continuous map $f : X \to X$ has a fixed point. For the only non-vanishing homology group is $H_0(K, \mathbb{Q}) \cong \mathbb{Q}$, and f_* must act as the identity on this – so $L(f) = 1$.

(b) Let $X = L_n$ be the n^{th} Lens space. Then $H_1(X, \mathbb{Q}) = 0$, so the only non-vanishing rational homology group is $H_0(X, \mathbb{Q}) \cong \mathbb{Q}$, and as before any continuous $f : X \to X$ has $L(f) = 1$.

(c) Let G be a topological group whose underlying space is a connected polyhedron, and assume $G \neq \{e\}$. Then $\chi(G) = 0$. For if g is any element of $G \setminus \{e\}$, the mapping $f(x) = g \cdot x$ has no fixed points, but if u is a path from e to g in G, then $H(x, y) = u(t) \cdot x$ gives a homotopy from id_G to f. So $L(f) = L(\mathrm{id}_G) = \chi(G)$.

(d) In particular, then torus $T = M_1$ is the only triangulable 2-manifold which admits a topological group structure. There are two 2-manifolds of $\chi = 0$, namely the torus M_1 and the Klein bottle N_2, but $\Pi_1(N_2)$ is non-abelian (example sheet 1, question 11) and Π_1 of any topological group must be abelian (example sheet 1, question 8). But T *is* a topological group, since it's $\mathbb{R}^2/\mathbb{Z}^2$.

Algebraic Topology Examples 1

PTJ Lent 2011

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1. Let $a\colon S^n \to S^n$ be the *antipodal map* (defined by $a(\mathbf{x}) = -\mathbf{x}$). Show that a is homotopic to the identity map if n is odd. [Hint: try $n = 1$ first! Later in the course, we'll be able to strengthen 'if' to 'if and only if'.]

2. Which of the capital letters $A, B, C, \ldots, Z$ are contractible? And which are homotopy equivalent to S^1?

3. Let $f\colon X \to Y$ be a continuous map, and suppose we are given (not necessarily equal) continuous maps $g, h\colon Y \rightrightarrows X$ such that $gf \simeq \mathrm{id}_X$ and $fh \simeq \mathrm{id}_Y$. Show that f is a homotopy equivalence.

4. (i) Let Y be the subspace $\{(x, 0) \mid x \in \mathbb{Q}, 0 \leq x \leq 1\}$ of $\mathbb{R}^2$, and let X be the *cone* on Y with vertex $(0, 1)$, i.e. the set of all points on straight line segments joining points of Y to $(0, 1)$. Show that X is contractible, but that in any homotopy H between the identity map on X and the constant map with value $(0, 0)$, the point $(0, 0)$ must 'move' (i.e. there exists t with $H((0, 0), t) \neq (0, 0)$).

*(ii) The problem in (i) arose because we chose the 'wrong' basepoint for X: if we had chosen $(0, 1)$ instead of $(0, 0)$, all would have been well. Can you find a contractible space Z such that *every* point of Z has to move in the course of a contracting homotopy?

5. Show that the torus minus a point, and the Klein bottle minus a point, are both homotopy equivalent to $S^1 \vee S^1$. [Hint: draw pictures showing how $S^1 \vee S^1$ can be embedded as a deformation retract in each space; do not attempt to write down precise formulae for the homotopies.]

6. Consider S^m embedded in S^n ($m < n$) as the subspace $\{(x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0) \mid \sum x_i^2 = 1\}$. Show that $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1}.

7. Let (X, x) and (Y, y) be two based spaces. Prove that $\Pi_1(X \times Y, (x, y)) \cong \Pi_1(X, x) \times \Pi_1(Y, y)$.

8. (i) Let A be a set equipped with two binary operations $\cdot$ and $*$, having a common (two-sided) identity element c and satisfying the 'interchange law'

$$(p \cdot q) * (r \cdot s) = (p * r) \cdot (q * s)$$

which says that each of the operations is a 'homomorphism' relative to the other. Show that the two operations coincide, and that they are (it is?) associative and commutative. [Hint: make appropriate substitutions in the interchange law. This piece of pure algebra is known as the *Eckmann–Hilton argument*: it has many applications besides the two described below.]

(ii) Let X be a space equipped with a continuous binary operation $m\colon X \times X \to X$ having a two-sided identity element e. Use part (i) and the previous question to show that $\Pi_1(X, e)$ is abelian. [Familiar examples of such spaces include topological groups; but the existence of inverses, and even the associativity of multiplication, are not needed for this result.]

*(iii) The *second homotopy group* $\Pi_2(X, x)$ of a pointed space (X, x) has elements which are homotopy classes of continuous maps from the unit square I^2 to X which map the boundary ∂I^2 to x (the homotopies between such maps being required to fix ∂I^2). Show that there are two possible ways ('horizontal' and 'vertical') of composing two such '2-dimensional loops', and deduce that $\Pi_2(X, x)$ is an abelian group. [For $n > 2$, $\Pi_n(X, x)$ is defined similarly using the unit n-cube I^n; it too is always abelian.]

9. Recall that, given a continuous map $f\colon S^{n-1} \to X$, we write $X \cup_f B^n$ for the space obtained by glueing an n-ball to X along f, i.e. the quotient of the disjoint union of X and B^n by the smallest equivalence relation which identifies $\mathbf{x}$ with $f(\mathbf{x})$ for each $\mathbf{x} \in S^{n-1}$. If f and g are homotopic maps $S^{n-1} \rightrightarrows X$, show that the spaces $X \cup_f B^n$ and $X \cup_g B^n$ are homotopy equivalent.

10. The 'topologist's dunce cap' D is the space obtained from the cone on the circle $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ with vertex $(0, 0, 1)$ by identifying the points $(\cos\, 2\pi t, \sin\, 2\pi t, 0)$ and $(1-t, 0, t)$ for $0 \le t \le 1$. Show that D is contractible. [Hint: use the previous question; it's helpful to 'flatten out' the cone by cutting it along the line $\{(1-t, 0, t) \mid 0 \le t \le 1\}$.]

11. Construct a covering projection $p\colon \mathbb{R}^2 \to K$ where K is the Klein bottle, and use it to show that $\Pi_1(K)$ is isomorphic to the group whose elements are pairs $(m, n) \in \mathbb{Z}^2$, with group operation given by

$$(m, n) * (p, q) = (m + (-1)^n p, n + q) \quad .$$

***12**. Let $p\colon X' \to X$ be a covering projection, and suppose given basepoints x, x' with $p(x') = x$. Show that, for any $n > 1$, p induces an isomorphism $\Pi_n(X', x') \cong \Pi_n(X, x)$ (for the definition of Π_n, see question 8(iii)). Deduce that $\Pi_n(S^1)$ is trivial for all $n > 1$. [Warning: this result does *not* generalize to higher-dimensional spheres: we have $\Pi_n(S^n) \cong \mathbb{Z}$ for all n, but $\Pi_m(S^n)$ can be nontrivial for $m > n$.]

***13**. Let X be an arbitrary metric space, and K a compact metric space. Given two continuous maps $f, g\colon K \rightrightarrows X$, explain why the function $k \mapsto d(f(k), g(k))$ (where d is the metric on X) is bounded and attains its bounds. Show also that

$$\overline{d}(f, g) = \sup\, \{d(f(k), g(k)) \mid k \in K\}$$

defines a metric on the set $\mathrm{Cts}(K, X)$ of all continuous maps $K \to X$.

Given $H\colon K \times I \to X$, show that H is continuous if and only if the function $\widehat{H}$ defined by $\widehat{H}(t)(k) = H(k, t)$ is a continuous function $I \to \mathrm{Cts}(K, X)$. Deduce that X is simply connected if and only if $\mathrm{Cts}(S^1, X)$ is path-connected.

***14**. A space X is said to be *locally path-connected* (or sometimes *semi-locally path-connected*) if, given any $x \in X$ and any open neighbourhood U of x, there exists a smaller open neighbourhood $V \subseteq U$ such that any two points of V may be joined by a path taking values in U. If X is a metric space, show that this condition is equivalent to saying that the mapping $f\colon \mathrm{Cts}(I, X) \to X$ sending a path u to $u(0)$ is an open map. (Recall that a map $g\colon Y \to Z$ between topological spaces is said to be *open* if $g(U)$ is open in Z whenever U is open in Y.)

At first sight, a more 'natural' definition of local path-connectedness would be to say that every open subset can be written as a union of path-connected open subsets (i.e., the path-connected opens form a base for the topology). Can you find an example of a space which fails to satisfy this condition but satisfies the one in the previous paragraph? [The counterexample from question 4(ii) might be helpful.]

Algebraic Topology Examples 2

PTJ Lent 2011

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1. Show that the covering projection $\mathbb{R}^2 \to K$ from question 11 on sheet 1 may be factored as $\mathbb{R}^2 \to T \to K$, where T is the 2-dimensional torus $S^1 \times S^1$. Identify the subgroup of index 2 in $\Pi_1(K)$, isomorphic to $\mathbb{Z} \times \mathbb{Z}$, corresponding to the covering $T \to K$.

2. Show that the free group F_2 on two generators has exactly three subgroups of index 2 [hint: consider homomorphisms $F_2 \to \mathbb{Z}/2\mathbb{Z}$]. Draw pictures of the three corresponding double coverings of $S^1 \vee S^1$, and calculate their fundamental groups.

3. Let X be the subspace of $\mathbb{R}^2$ which is the union of the x-axis with $\bigcup_{n\in\mathbb{Z}} C_n$, where C_n is the circle with centre $(n, \frac{1}{3})$ and radius $\frac{1}{3}$. Construct a covering projection $X \to S^1 \vee S^1$. Show that X is homotopy equivalent to a countably-infinite wedge union of circles, and deduce that the free group on two generators contains a subgroup which is free on countably many generators.

4. Let X be a Hausdorff topological space, and let G be a finite subgroup of the group of all homeomorphisms $X \to X$, such that no member of G other than the identity has a fixed point. Let X/G denote the set of G-orbits, topologized as a quotient space of X. Show that the quotient map $X \to X/G$ is a covering projection, and deduce that if X is simply connected and locally path-connected then $\Pi_1(X/G) \cong G$. Hence show that, for any odd $n > 1$ and any $m > 1$, there is a quotient space of S^n with fundamental group $\mathbb{Z}/m\mathbb{Z}$. [Hint: regard $\mathbb{R}^{2k}$ as $\mathbb{C}^k$. In contrast, we'll see later that for even n, the only group which can act on S^n so that no non-identity element has a fixed point is the cyclic group of order 2.]

5. Let X be the subspace

$$\{(x, \sin\ \pi/x) \mid 0 \le x \le 1\} \cup \{(0, y) \mid -1 \le y \le 1\}$$

of $\mathbb{R}^2$, and let Y be the union of X with the three line segments $\{(0, y) \mid 1 \le y \le 2\}$, $\{(x, 2) \mid 0 \le x \le 1\}$ and $\{(1, y) \mid 0 \le y \le 2\}$.

(i) Show that X is connected but not path-connected (if you didn't already do this in Metric and Topological Spaces).

(ii) Show that Y is simply connected but not locally path-connected.

(iii) Show that there is a double covering $p\colon Z \to Y$ where Z is connected (and thus not homeomorphic to $Y \times \{1, 2\}$).

6. (i) Let H be the Hawaiian earring $\bigcup_{n=1}^{\infty} C_n \subseteq \mathbb{R}^2$, where C_n is the circle with centre $(0, -\frac{1}{n})$ and radius $\frac{1}{n}$. Show that $\Pi_1(H, (0, 0))$ is uncountable, and deduce that it is not finitely presented. [In showing that you have constructed uncountably many distinct elements of $\Pi_1(H)$, you may find it helpful to consider the continuous maps $H \to S^1$ collapsing all but one of the circles in H to a point, and wrapping the remaining one around S^1.]

*(ii) Let H' be the reflection of H in the x-axis. Is $\Pi_1(H \cup H', (0, 0))$ isomorphic to the free product of two copies of $\Pi_1(H, (0, 0))$?

*(iii) Now regard H and H' as embedded in the plane $\{(x, y, z) \mid z = 0\} \subseteq \mathbb{R}^3$; let C be the cone on H with vertex $(0, 0, 1)$, and C' the cone on H' with vertex $(0, 0, -1)$. Is $C \cup C'$ simply connected?

7. Let $f\colon S^1 \to X$ be a continuous map, and consider the space $Y = X \cup_f B^2$, defined as in question 9 on sheet 1. Let $x = f(1)$; show that $\Pi_1(Y, x) \cong \Pi_1(X, x)/N$, where N is the normal subgroup generated by $f_*(g)$ for a generator g of $\Pi_1(S^1)$. Deduce that, for any finitely presented group G, there is a compact path-connected space Z with $\Pi_1(Z) \cong G$.

8. Show that the Klein bottle K may be described as $(S^1 \vee S^1) \cup_f B^2$ for a suitable map $f\colon S^1 \to S^1 \vee S^1$. Use question 7 to give a presentation of $\Pi_1(K)$ with two generators and one relation, and verify directly that this group is isomorphic to the one described in question 11 on sheet 1.

9. Show that the finitely presented groups

$$G = \langle a, b \mid a^3 = b^2 \rangle \qquad \text{and} \qquad H = \langle x, y \mid xyx = yxy \rangle$$

are isomorphic. Show also that this group is non-abelian and infinite. [Hint: find surjective homomorphisms to the symmetric group S_3 and to $\mathbb{Z}$.]

10. Complex projective space $\mathbb{C}P^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation which identifies $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{x} = t\mathbf{y}$ for some (complex) scalar t. Show that

(i) there is a quotient map $h_n\colon S^{2n+1} \to \mathbb{C}P^n$ such that the inverse image of each point is a copy of S^1;

(ii) for $n > 1$, $\mathbb{C}P^n$ is homeomorphic to $\mathbb{C}P^{n-1} \cup_{h_{n-1}} B^{2n}$;

(iii) $\mathbb{C}P^1$ is homeomorphic to S^2.

Deduce that $\mathbb{C}P^n$ is simply connected for all n.

***11**. Let G be a connected graph, considered as a topological space in the way that we did for the Cayley graph of F_2 in lectures. Show that there is a simply connected subgraph G' containing all the vertices of G, and deduce that $\Pi_1(G)$ is isomorphic to the free group generated by the edges of $G \setminus G'$. Hence show (generalizing the result of question 3) that any subgroup of a free group is free.

***12**. (i) Let $G = SU(2)$ be the group of 2×2 unitary matrices with determinant 1. Show that (the underlying space of) G is homeomorphic to S^3.

(ii) Let $H = SO(3)$, the group of 3×3 orthogonal matrices with determinant 1. Show that H is homeomorphic to $\mathbb{R}P^3$. [Hint: first show that the set of 180° rotations is homeomorphic to $\mathbb{R}P^2$.]

(iii) Show that $\{\pm I\}$ is a normal subgroup of $SU(2)$, and that the quotient $SU(2)/\{\pm I\}$ is isomorphic (as well as homeomorphic) to $SO(3)$.

(iv) Show that there is a quotient space X of S^3 such that $\Pi_1(X)$ is a non-abelian group of order 120, whose only nontrivial normal subgroup has order 2. [Use question 4: recall that the group of rotational symmetries of a regular dodecahedron is isomorphic to A_5.]

***13**. View S^3 as the set $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$, and let $T \subseteq S^3$ be the set

$$\{(z, w) \mid z^2 = w^3\} \ .$$

(T is called the *trefoil knot*.)

(i) Show that T is homeomorphic to S^1.

(ii) Use the Seifert–Van Kampen theorem to show that $\Pi_1(S^3 \setminus T)$ is isomorphic to the group G in question 9. [Method: let $X = \{(z, w) \mid |z|^2 = |w|^3\}$. Show that $S^3 \setminus X$ is homeomorphic to the disjoint union of two copies of $S^1 \times B^2$, and that $X \setminus T$ is homeomorphic to $S^1 \times (0, 1)$.]

(iii) Let $U = \{(z, 0) \mid |z| = 1\}$ be the *unknot* in S^3. Show that there is no homeomorphism $f\colon S^3 \to S^3$ for which $f(U) = T$. [Recall question 6 on sheet 1.]

Algebraic Topology Examples 3

PTJ Lent 2011

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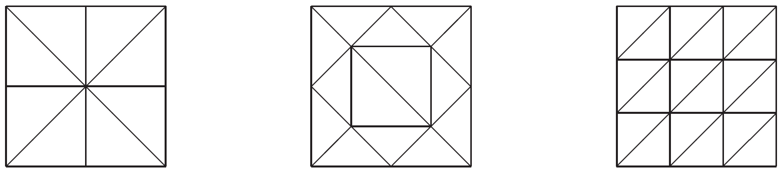
1. (i) If we regard the projective plane $\mathbb{R}P^2$ as the quotient of the unit square obtained by identifying opposite points around its boundary, which (if any) of the following pictures represents a triangulation of $\mathbb{R}P^2$?



*(ii) What is the smallest possible number of 2-simplices in a triangulation of $\mathbb{R}P^2$?

2. Show that it is possible to choose an infinite sequence of points $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)$ in $\mathbb{R}^m$ which are *in general position* in the sense that no $m+1$ of them lie in a proper affine subspace (i.e., a coset of a proper vector subspace). Deduce that if K is an abstract simplicial complex having no simplices of dimension greater than n, it is possible to find a *geometric realization* of K (that is, a space having a triangulation isomorphic to K) which is a subspace of $\mathbb{R}^{2n+1}$. [This result is best possible; it can be shown that the n-skeleton of a $(2n+2)$-simplex cannot be realized in $\mathbb{R}^{2n}$.]

3. Use the Simplicial Approximation Theorem to show

(i) that if X and Y are polyhedra then there are only countably many homotopy classes of continuous maps $X \to Y$;

(ii) that if $m < n$ then any continuous map $S^m \to S^n$ is homotopic to a constant map.

4. Show that the fundamental group of a polyhedron depends only on its 2-skeleton: that is, for any simplicial complex K and vertex a of k, we have $\Pi_1(|K|, a) \cong \Pi_1(|K_{(2)}|, a)$, where $K_{(2)}$ is the 2-skeleton of K. [Apply the Simplicial Approximation Theorem to paths in $|K|$ and homotopies between them. Question 10 below provides an explicit combinatorial method of calculating the fundamental group of a polyhedron based on this idea.]

5. Let K and L be simplical complexes. Construct a triangulation of $|K| \times |L|$. [Method: take the points $(\hat{\sigma}, \hat{\tau})$, where σ is a simplex of K and τ is a simplex of L, as vertices, and say that a sequence

$$((\widehat{\sigma_0}, \widehat{\tau_0}), (\widehat{\sigma_1}, \widehat{\tau_1}), \ldots, (\widehat{\sigma_n}, \widehat{\tau_n}))$$

spans a simplex if and only if we have $\sigma_{i-1} \leq \sigma_i$ and $\tau_{i-1} \leq \tau_i$ for each i, at least one of the two inequalities being proper. You may find it helpful to consider first what this gives when $|K|$ and $|L|$ are both 1-simplices.]

6. Two simplicial maps $h, k : K \rightrightarrows L$ are said to be *contiguous* if, for each simplex σ of K, there is a simplex $\sigma*$ of L such that both $h(\sigma)$ and $k(\sigma)$ are faces of $\sigma*$. Show that

(i) any two simplicial approximations to a given map $f : |K| \to |L|$ are contiguous;

(ii) any two contiguous maps $K \rightrightarrows L$ induce homotopic maps $|K| \rightrightarrows |L|$;

(iii) if $f, g : |K| \rightrightarrows |L|$ are any two homotopic maps, then for a suitable subdivision $K^{(N)}$ of K there exists a sequence of simplicial maps $h_1, \ldots, h_k : K^{(N)} \to L$ such that h_1 is a simplicial approximation to f, h_n is a simplicial approximation to g and each pair (h_i, h_{i+1}) is contiguous. [Method: let H be a homotopy between f and g, and show that for sufficiently large n and N the mappings $x \mapsto H(x, \frac{i-1}{n})$ and $x \mapsto H(x, \frac{i}{n})$ have a common simplicial approximation defined on $K^{(N)}$, for each $i \leq n$.]

7. Let K be a simplicial complex, and let CK be the cone on K, i.e. the complex obtained from K by adding one new vertex v, and adding new simplices spanned by v together with the vertices of σ, for each simplex σ of K. Show that the inclusion $\{v\} \to CK$ induces a chain homotopy equivalence $C_*(\{v\}) \to C_*(CK)$.

8. Let X be the 'complete n-simplex', i.e. the simplicial complex formed by an n-simplex together with all its faces. What is the rank of the chain group $C_k(X)$? Show that, for $1 \leq k \leq n$, the kth homology group of the k-skeleton $X_{(k)}$ of X is free abelian of rank $\binom{n}{k+1}$. [Hint: calculate the ranks of the groups $Z_k(X)$ for all k, using question 7 to show that $H_k(X) = 0$ for all $k > 0$.]

9. Let K be a simplicial complex satisfying the following conditions:
(i) K has no simplices of dimension greater than n;
(ii) Every $(n-1)$-simplex of K is a face of exactly two n-simplices;
(iii) For any two n-simplices σ and τ of K, there exists a finite sequence of n-simplices, beginning with σ and ending with τ, in which each adjacent pair of simplices have a common $(n-1)$-dimensional face.
Show that $H_n(K)$ is either $\mathbb{Z}$ or the trivial group, and that in the former case it is generated by a cycle with is the sum of all the n-simplices of K, with suitable orientations. Give examples with $n = 2$ to show that either possibility can occur.

***10**. Let K be a simplicial complex. We define an *edge path* in K to be a finite sequence $(a_0, a_1, \ldots, a_n)$ of vertices of K such that (a_i, a_{i+1}) spans a simplex for each i. An *edge loop* is an edge path such that $a_0 = a_n$; the *product* of two edge paths $(a_0, \ldots, a_n)$ and $(a_n, \ldots, a_m)$ is $(a_0, \ldots, a_n, \ldots, a_m)$. Two edge paths are said to be *equivalent* if one can be converted into the other by a finite sequence of moves of the form: replace $(\ldots, a_i, a_{i+1}, a_{i+2}, \ldots)$ by $(\ldots, a_i, a_{i+2}, \ldots)$ provided $\{a_i, a_{i+1}, a_{i+2}\}$ spans a simplex of K (or the inverse of this move). (We allow the possibility that a_i, a_{i+1}, a_{i+2} may not all be distinct; thus, for example, we may always replace $(\ldots, a_i, a_{i+1}, a_i, \ldots)$ by $(\ldots, a_i, a_i, \ldots)$, and we may further replace this by $(\ldots, a_i, \ldots)$ provided there is at least one other vertex in the sequence.) Show that equivalence classes of edge loops based at a_0 form a group $E(K, a_0)$, and use the Simplicial Approximation Theorem (plus question 6) to show that $E(K, a_0) \cong \Pi_1(|K|, a_0)$.

***11**. Let K be a simplicial complex, and a a vertex of K. Show that there is a homomorphism $h : \Pi_1(|K|, a) \to H_1(K)$. [$h$ is called the *Hurewicz homomorphism*; to construct it, observe that an edge path in K, as defined in the previous question, can be thought of as an 'ordered sum' of oriented 1-simplices, whereas a 1-chain is an unordered sum of such simplices.] Show also that h is surjective if K is connected; and that if $f : K \to L$ is a simplicial map sending a to b then the diagram

$$\begin{tikzcd} \Pi_1(|K|, a) \arrow[r, "h"] \arrow[d, "f_*"] & H_1(K) \arrow[d, "f_*"] \\ \Pi_1(|L|, b) \arrow[r, "h"] & H_1(L) \end{tikzcd}$$

commutes. [It can be shown that the kernel of h is exactly the commutator subgroup of $\Pi_1(|K|)$ — that is, for connected K, $H_1(K)$ is isomorphic to the largest abelian quotient group of $\Pi_1(|K|)$.]

***12**. Consider the quotient space X of S^3 constructed in question 12(iv) on sheet 2. Use the previous question to show that $H_1(X) = 0$. [It can be shown that X has the same homology groups as S^3, but it is not homotopy equivalent to S^3.]

Algebraic Topology Examples 4

PTJ Lent 2011

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1. For each of the following exact sequences of abelian groups and homomorphisms, say as much as you can about the unknown group G and/or the unknown homomorphism α:
(i) $0 \to \mathbb{Z}/2\mathbb{Z} \to G \to \mathbb{Z} \to 0$;
(ii) $0 \to \mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$;
(iii) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to 0$;
(iv) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$;
(v) $0 \to \mathbb{Z}/3\mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$.

2. Use the Mayer–Vietoris theorem to calculate the homology groups of the following spaces. [You may assume that suitable triangulations exist in each case.]
(i) The Klein bottle K, regarded as the space obtained by glueing together two copies of $S^1 \times I$.
(ii) The space X obtained by removing the interior of a small disc from a torus. [Recall question 5 on sheet 1.]
(iii) The space Y obtained from the space X of part (ii) and a Möbius band M by identifying the boundary of M with the edge of the 'hole' in X.

3. By restricting the (evident) homeomorphism $B^{r+s+2} \cong B^{r+1} \times B^{s+1}$ to the boundaries of these two spaces, and assuming the existence of suitable triangulations, show that we can triangulate S^{r+s+1} as the union of two subcomplexes L and M, where $|L| \simeq S^r$, $|M| \simeq S^s$ and $|L \cap M| \cong S^r \times S^s$. Use this to calculate the homology groups of $S^r \times S^s$ for $r, s \geq 1$. [Distinguish between the cases $r = s$ and $r \neq s$.]

4. Let A be a 2×2 matrix with integer entries. Show that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ represented by A respects the equivalence relation $\sim$ on $\mathbb{R}^2$ given by $(x, y) \sim (z, w)$ iff $x - z$ and $y - w$ are integers, and deduce that it induces a continuous map f_A from the torus T to itself. Calculate the effect of f_A on the homology groups of T, and show in particular that f_A is a homeomorphism if and only if it induces an isomorphism $H_2(T) \to H_2(T)$. [It can be shown that every continuous map $T \to T$ is homotopic to f_A for some A.]

5. Let K be a (geometric) simplicial complex in $\mathbb{R}^m$. The *suspension* SK of K is the complex in $\mathbb{R}^{m+1}$ whose vertices are those of K (regarded as lying in $\mathbb{R}^m \times \{0\}$) and the two points $(0, \ldots, 0, \pm 1)$, and whose simplices are those of K together with those spanned by the vertices of a simplex of K together with one or the other (but not both) of the new vertices.
(i) Verify that SK is a simplicial complex, and show in particular that if $|K| \cong S^n$ then $|SK| \cong S^{n+1}$.
(ii) Use the Mayer–Vietoris theorem to show that $H_r(SK) \cong H_{r-1}(K)$ for $r \geq 2$, and that $H_1(SK) = 0$ if K is connected.
(iii) Let $f : K \to K$ be a simplicial map, and let $\widetilde{f} : SK \to SK$ be the unique extension of f to a simplicial map which interchanges the two vertices $(0, \ldots, 0, \pm 1)$. Show that, if we identify $H_r(SK)$ with $H_{r-1}(K)$, then $\widetilde{f}_* : H_r(SK) \to H_r(SK)$ sends a homology class c to $-f_*(c)$.
(iv) Deduce that if $a : S^n \to S^n$ is the antipodal map, then $a_* : H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$. [Compare question 1 on sheet 1.]

6. Let $n \geq 2$. Show that the following statements are equivalent:
(i) There is no continuous map $f : S^n \to S^{n-1}$ satisfying $f(-\mathbf{x}) = -f(\mathbf{x})$ for all $\mathbf{x}$.
(ii) There is no continuous map $B^n \to S^{n-1}$ whose restriction to S^{n-1} is the antipodal map.

(iii) For every continuous map $g\colon \mathbb{R}P^n \to \mathbb{R}P^{n-1}$, $g_*\colon \Pi_1(\mathbb{R}P^n) \to \Pi_1(\mathbb{R}P^{n-1})$ is the trivial homomorphism.

(iv) For every continuous map $h\colon S^n \to \mathbb{R}^n$, there exists $\mathbf{x} \in S^n$ such that $h(\mathbf{x}) = h(-\mathbf{x})$.

(v) For any decomposition of S^n as the union of $n+1$ closed subsets $F_1, \ldots, F_{n+1}$, there exists $\mathbf{x} \in S^n$ such that $\mathbf{x}$ and $-\mathbf{x}$ belong to the same set F_i.

[Hint: show (ii) $\Leftrightarrow$ (i) $\Leftrightarrow$ (iii) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). For (i) $\Rightarrow$ (iv), given a counterexample to (iv), consider the mapping $\mathbf{x} \mapsto (h(\mathbf{x}) - h(-\mathbf{x}))/\|h(\mathbf{x}) - h(-\mathbf{x})\|$. For (iv) $\Rightarrow$ (v), consider the mapping $\mathbf{x} \mapsto (d(\mathbf{x}, F_1), \ldots, d(\mathbf{x}, F_n))$, where $d(\mathbf{x}, F) = \inf\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in F\}$.]

Use fundamental groups to show that (iii) is true for $n = 2$, and homology groups to show that (ii) is true for all n. [(iv) is known as the *Borsuk–Ulam Theorem*, and (v) is known as the *Lyusternik–Shnirel'man Theorem*.]

7. Suppose that a simplicial complex K is the union of subcomplexes L and M, and that P is the union of subcomplexes Q and R. Suppose further that $f\colon K \to P$ is a simplicial map which maps L into Q and M into R. Show that there is a commutative diagram

$$\begin{tikzcd}
\cdots & H_r(L \cap M) \arrow[r] \arrow[d, "f_*"] & H_r(L) \oplus H_r(M) \arrow[r] \arrow[d, "f_*"] & H_r(K) \arrow[r] \arrow[d, "f_*"] & H_{r-1}(L \cap M) \arrow[d, "f_*"] & \cdots \\
\cdots & H_r(Q \cap R) \arrow[r] & H_r(Q) \oplus H_r(R) \arrow[r] & H_r(P) \arrow[r] & H_{r-1}(Q \cap R) & \cdots
\end{tikzcd}$$

in which the rows are Mayer–Vietoris sequences.

8. By considering S^n as the union of the subsets given by the inequalities $|x_{n+1}| \leq \frac{1}{2}$ and $|x_{n+1}| \geq \frac{1}{2}$, and using the results of questions 5 and 7, show that the homology groups of real projective space $\mathbb{R}P^n$ are given by

$$\begin{aligned}
H_r(\mathbb{R}P^n) &\cong \mathbb{Z} && \text{if } r = 0, \text{ or if } r = n \text{ and } n \text{ is odd} \\
&\cong \mathbb{Z}/2\mathbb{Z} && \text{if } r \text{ is odd and } 0 < r < n \\
&= 0 && \text{if } r > n, \text{ or if } 0 < r \leq n \text{ and } r \text{ is even.}
\end{aligned}$$

[You may assume the existence of suitable triangulations.]

9. Use the results of question 10 on sheet 2 to compute the homology groups of $\mathbb{C}P^n$. [Perhaps surprisingly, this turns out to be much easier than $\mathbb{R}P^n$.]

***10**. By a *knot* in S^3, we mean an embedding $f\colon S^1 \to S^3$ — that is, a homeomorphism from S^1 to a subspace of S^3. We say the knot is *tame* if f can be 'thickened up' to an embedding $\widetilde{f}\colon S^1 \times B^2 \to S^3$, with $f = \widetilde{f}|_{S^1 \times \{0\}}$. Assuming (as always!) the existence of suitable triangulations, show that if f is a tame knot, then the space obtained by removing the interior of the image of $\widetilde{f}$ from S^3 has the same homology groups as S^1. [Thus homology groups, unlike the fundamental group (see question 13 on sheet 2), are of no use for distinguishing between different knots.]

***11**. Let $n \geq 2$ be an even integer. Show that if $f\colon S^n \to S^n$ is a continuous mapping without fixed points, then $f_*\colon H_n(S^n) \to H_n(S^n)$ is multiplication by -1. Deduce that if $p\colon S^n \to X$ is a nontrivial covering projection then $\Pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$. [Compare question 4 on sheet 2. Remarkably, such an X need not be homeomorphic to $\mathbb{R}P^n$; there is a counterexample with $n = 4$.]

***12**. Are the three stainless steel sculptures along the Clarkson Road front of the Isaac Newton Institute topologically equivalent? (That is, are there homeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$ mapping each of them onto the other two?)

Local Path-Connectedness — An Apology

PTJ Lent 2011

For around 40 years I have believed that the two possible definitions of local path-connectedness, as set out in question 14 on the first Algebraic Topology example sheet, are not equivalent. This belief has been reinforced by the many topology textbooks which insist that the first, less 'natural', definition is the right one to use; not a few of them further reinforce the impression by calling it 'semi-local path-connectedness', which is clearly meant to imply that it is strictly weaker than the 'natural' definition.

Some 25 years ago, I discovered the equivalence of the formally weaker definition with the openness of the canonical map $\mathrm{Cts}(I, X) \to X \times X$ sending u to $(u(0), u(1))$; for reasons which I don't need to go into here, that seemed to me a good and sufficient explanation of why it was the right definition to use (presuming the two to be inequivalent). At the time I wondered at the fact that no-one had bothered to explain this reason to me when I was a student; and I resolved that I would be more honest with my students if I ever found myself lecturing on algebraic topology in future.

Therefore, when preparing example sheets at the start of this term, I initially put it on example sheet 1 as (the first half of) question 14. I then thought that I could make the question a bit easier by replacing $u \mapsto (u(0), u(1))$ by the mapping $u \mapsto u(0)$, so I changed it. Unfortunately, whilst this does make the implication in one direction easier to prove, it makes the converse false, as can be seen by considering a space X which is totally disconnected but not discrete (for example, $X = \mathbb{Q}$).

It also occurred to me that I didn't actually know a counterexample to separate the two definitions (and none of the textbooks that I then consulted provided one). I though about it for a while, and succeeded in convincing myself that a modification of the counterexample that I knew for question 4(ii) would do the trick; so I added the second half of question 14.

More recently, I happened to look up something in the textbook *Topology* (Ellis Horwood, 1988) by my old friend Ronnie Brown, who was for many years Professor of Pure Maths at Bangor in north Wales, but is now retired. (Ronnie is a lovable eccentric: I wouldn't recommend his textbook to anyone other than a high first-class student, since it's likely to be far more confusing than helpful to anyone else.) I happened to notice his definition of local path-connectedness: he dutifully gives the 'unnatural' definition first, but then immediately proves that it is equivalent to the 'natural' one. Moreover, the proof of equivalence is carried over unchanged from the first edition of Ronnie's book, published in 1968 by McGraw–Hill. **So why didn't anyone tell me that 40 years ago?**

Here, with my sincere apologies to anyone who may have wasted time looking for a counterexample, is Ronnie's proof. Suppose X satisfies the 'weaker' definition of local path-connectedness. Let $x \in X$, and let U be an open neighbourhood of x. Let U' be the path-component of x in U (i.e., the set of points which can be joined to x by paths in U). If $y \in U'$, then U is an open neighbourhood of y, so by the 'weaker' definition there exists an open V with $y \in V \subseteq U$ such that y can be joined to any point of V by a path in U. Since we can paste these paths on to a path from x to y, we clearly have $V \subseteq U'$; so U' is a neighbourhood of each of its points, i.e. it is open. And U' is path-connected by definition; so the path-connected open sets form a base for the topology.