Paper 1, Section I
1C Vectors and Matrices

For $z, a \in \mathbb{C}$ define the *principal value* of $\log z$ and hence of z^a. Hence find all solutions to

(i) $z^{\mathrm{i}} = 1$

(ii) $z^{\mathrm{i}} + \overline{z}^{\,\mathrm{i}} = 2\mathrm{i}\,,$

and sketch the curve $|z^{\mathrm{i}+1}| = 1$.

Paper 1, Section I
2A Vectors and Matrices

The matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad , \qquad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad .$$

Find the matrix A' that represents Φ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \quad , \qquad C' = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \quad .$$

Paper 1, Section II

5C Vectors and Matrices

Explain why each of the equations

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} \qquad (1)$$

$$\mathbf{x} \times \mathbf{c} = \mathbf{d} \qquad (2)$$

describes a straight line, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constant vectors in $\mathbb{R}^3$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero, $\mathbf{c} \cdot \mathbf{d} = 0$ and λ is a real parameter. Describe the geometrical relationship of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ to the relevant line, assuming that $\mathbf{d} \neq \mathbf{0}$.

Show that the solutions of (2) satisfy an equation of the form (1), defining $\mathbf{a}$, $\mathbf{b}$ and $\lambda(\mathbf{x})$ in terms of $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{a} \cdot \mathbf{b} = 0$ and $|\mathbf{b}| = |\mathbf{c}|$. Deduce that the conditions on $\mathbf{c}$ and $\mathbf{d}$ are sufficient for (2) to have solutions.

For each of the lines described by (1) and (2), find the point $\mathbf{x}$ that is closest to a given fixed point $\mathbf{y}$.

Find the line of intersection of the two planes $\mathbf{x} \cdot \mathbf{m} = \mu$ and $\mathbf{x} \cdot \mathbf{n} = \nu$, where $\mathbf{m}$ and $\mathbf{n}$ are constant unit vectors, $\mathbf{m} \times \mathbf{n} \neq \mathbf{0}$ and μ and ν are constants. Express your answer in each of the forms (1) and (2), giving both $\mathbf{a}$ and $\mathbf{d}$ as linear combinations of $\mathbf{m}$ and $\mathbf{n}$.

Paper 1, Section II

6A Vectors and Matrices

The map $\Phi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n}) + \alpha(\mathbf{n} \cdot \mathbf{x})\,\mathbf{n}$ is defined for $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$ and α is a constant.

(a) Find the inverse map Φ^{-1}, when it exists, and determine the values of α for which it does.

(b) When Φ is not invertible, find its image and kernel, and explain geometrically why these subspaces are perpendicular.

(c) Let $\mathbf{y} = \Phi(\mathbf{x})$. Find the components A_{ij} of the matrix A such that $y_i = A_{ij}x_j$. When Φ is invertible, find the components of the matrix B such that $x_i = B_{ij}y_j$.

(d) Now let A be as defined in (c) for the case $\mathbf{n} = \frac{1}{\sqrt{3}}(1,1,1)$, and let

$$C \;=\; \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}.$$

By analysing a suitable determinant, for all values of α find all vectors $\mathbf{x}$ such that $A\mathbf{x} = C\mathbf{x}$. Explain your results by interpreting A and C geometrically.

Paper 1, Section II
7B Vectors and Matrices

(a) Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -2 & 3 \end{pmatrix} .$$

(b) Under what conditions on the $3{\times}3$ matrix A and the vector $\mathbf{b}$ in $\mathbb{R}^3$ does the equation

$$A\mathbf{x} = \mathbf{b} \qquad (*)$$

have 0, 1, or infinitely many solutions for the vector $\mathbf{x}$ in $\mathbb{R}^3$? Give clear, concise arguments to support your answer, explaining why just these three possibilities are allowed.

(c) Using the results of (a), or otherwise, find all solutions to $(*)$ when

$$A = M - \lambda I \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

in each of the cases $\lambda = 0,\ 1,\ 2$.

Paper 1, Section II
8B Vectors and Matrices

(a) Let M be a real symmetric $n{\times}n$ matrix. Prove the following.

(i) Each eigenvalue of M is real.

(ii) Each eigenvector can be chosen to be real.

(iii) Eigenvectors with different eigenvalues are orthogonal.

(b) Let A be a real antisymmetric $n{\times}n$ matrix. Prove that each eigenvalue of A^2 is real and is less than or equal to zero.

If $-\lambda^2$ and $-\mu^2$ are distinct, non-zero eigenvalues of A^2, show that there exist orthonormal vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{w}$, $\mathbf{w}'$ with

$$A\mathbf{u} = \lambda\mathbf{u}' , \qquad A\mathbf{w} = \mu\mathbf{w}' ,$$
$$A\mathbf{u}' = -\lambda\mathbf{u} , \qquad A\mathbf{w}' = -\mu\mathbf{w} .$$