Paper 1, Section I
1A Vectors and Matrices

Let A be the matrix representing a linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ of $\mathbb{R}^m$, so that $\Phi(\mathbf{b}_i) = A_{ji}\mathbf{c}_j$. Let $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ be another basis of $\mathbb{R}^n$ and let $\{\mathbf{c}'_1, \ldots, \mathbf{c}'_m\}$ be another basis of $\mathbb{R}^m$. Show that the matrix A' representing Φ with respect to these new bases satisfies $A' = C^{-1}A\,B$ with matrices B and C which should be defined.

Paper 1, Section I
2C Vectors and Matrices

(a) The complex numbers z_1 and z_2 satisfy the equations

$$z_1^3 = 1\,, \qquad z_2^9 = 512\,.$$

What are the possible values of $|z_1 - z_2|$? Justify your answer.

(b) Show that $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for all complex numbers z_1 and z_2 . Does the inequality $|z_1 + z_2| + |z_1 - z_2| \leqslant 2\max(|z_1|, |z_2|)$ hold for all complex numbers z_1 and z_2? Justify your answer with a proof or a counterexample.

Paper 1, Section II
5A Vectors and Matrices

Let A and B be real $n \times n$ matrices.

(i) Define the trace of A, $\operatorname{tr}(A)$, and show that $\operatorname{tr}\left(A^T B\right) = \operatorname{tr}\left(B^T A\right)$.

(ii) Show that $\operatorname{tr}\left(A^T A\right) \geqslant 0$, with $\operatorname{tr}\left(A^T A\right) = 0$ if and only if A is the zero matrix. Hence show that

$$\left(\operatorname{tr}(A^T B)\right)^2 \leqslant \operatorname{tr}\left(A^T A\right) \operatorname{tr}\left(B^T B\right)\ .$$

Under what condition on A and B is equality achieved?

(iii) Find a basis for the subspace of 2×2 matrices X such that

$$\operatorname{tr}\left(A^T X\right) = \operatorname{tr}\left(B^T X\right) = \operatorname{tr}\left(C^T X\right) = 0\,,$$

$$\text{where} \qquad A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Paper 1, Section II

6C Vectors and Matrices

Let $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ be vectors in $\mathbb{R}^3$. Give a definition of the dot product, $\mathbf{a}_1 \cdot \mathbf{a}_2$, the cross product, $\mathbf{a}_1 \times \mathbf{a}_2$, and the triple product, $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$. Explain what it means to say that the three vectors are *linearly independent*.

Let $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ be vectors in $\mathbb{R}^3$. Let S be a 3×3 matrix with entries $S_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$. Show that

$$(\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3)(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3) \;=\; \det(S)\,.$$

Hence show that S is of maximal rank if and only if the sets of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are both linearly independent.

Now let $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ be sets of vectors in $\mathbb{R}^n$, and let T be an $n \times n$ matrix with entries $T_{ij} = \mathbf{c}_i \cdot \mathbf{d}_j$. Is it the case that T is of maximal rank if and only if the sets of vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ are both linearly independent? Justify your answer with a proof or a counterexample.

Given an integer $n > 2$, is it always possible to find a set of vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ in $\mathbb{R}^n$ with the property that every pair is linearly independent and that every triple is linearly dependent? Justify your answer.

Paper 1, Section II

7B Vectors and Matrices

Let A be a complex $n \times n$ matrix with an eigenvalue λ. Show directly from the definitions that:

(i) A^r has an eigenvalue λ^r for any integer $r \geqslant 1$; and

(ii) if A is invertible then $\lambda \neq 0$ and A^{-1} has an eigenvalue λ^{-1}.

For any complex $n \times n$ matrix A, let $\chi_A(t) = \det(A - tI)$. Using standard properties of determinants, show that:

(iii) $\chi_{A^2}(t^2) = \chi_A(t)\, \chi_A(-t)$; and

(iv) if A is invertible,

$$\chi_{A^{-1}}(t) = (\det A)^{-1}(-1)^n t^n \chi_A(t^{-1}) .$$

Explain, including justifications, the relationship between the eigenvalues of A and the polynomial $\chi_A(t)$.

If A^4 has an eigenvalue μ, does it follow that A has an eigenvalue λ with $\lambda^4 = \mu$? Give a proof or counterexample.

Paper 1, Section II

8B Vectors and Matrices

Let R be a real orthogonal 3×3 matrix with a real eigenvalue λ corresponding to some real eigenvector. Show algebraically that $\lambda = \pm1$ and interpret this result geometrically.

Each of the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

has an eigenvalue $\lambda = 1$. Confirm this by finding as many independent eigenvectors as possible with this eigenvalue, for each matrix in turn.

Show that one of the matrices above represents a rotation, and find the axis and angle of rotation. Which of the other matrices represents a reflection, and why?

State, with brief explanations, whether the matrices M, N, P are diagonalisable (i) over the real numbers; (ii) over the complex numbers.