Paper 1, Section I

1C Vectors and Matrices

Describe geometrically the three sets of points defined by the following equations in the complex z plane:

(a) $z\overline{\alpha} + \overline{z}\alpha = 0$, where α is non-zero;

(b) $2|z - a| = z + \overline{z} + 2a$, where a is real and non-zero;

(c) $\log z = \mathrm{i} \log \overline{z}$.

Paper 1, Section I

2B Vectors and Matrices

Define the Hermitian conjugate $A^{\dagger}$ of an $n \times n$ complex matrix A. State the conditions (i) for A to be Hermitian (ii) for A to be unitary.

In the following, A, B, C and D are $n \times n$ complex matrices and $\mathbf{x}$ is a complex n-vector. A matrix N is defined to be *normal* if $N^{\dagger}N = NN^{\dagger}$.

(a) Let A be nonsingular. Show that $B = A^{-1}A^{\dagger}$ is unitary if and only if A is normal.

(b) Let C be normal. Show that $|C\mathbf{x}| = 0$ if and only if $|C^{\dagger}\mathbf{x}| = 0$.

(c) Let D be normal. Deduce from (b) that if $\mathbf{e}$ is an eigenvector of D with eigenvalue λ then $\mathbf{e}$ is also an eigenvector of $D^{\dagger}$ and find the corresponding eigenvalue.

Paper 1, Section II

5C Vectors and Matrices

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors. By using suffix notation, prove that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}) = \mathbf{b}\cdot\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c}) \tag{1}$$

and

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}) = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a}\ . \tag{2}$$

The three distinct points A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ lie on the surface of the unit sphere centred on the origin O. The *spherical distance* between the points A and B, denoted $\delta(A,B)$, is the length of the (shorter) arc of the circle with centre O passing through A and B. Show that

$$\cos\delta(A,B) = \mathbf{a}\cdot\mathbf{b}\ .$$

A *spherical triangle* with vertices A, B, C is a region on the sphere bounded by the three circular arcs AB, BC, CA. The interior angles of a spherical triangle at the vertices A, B, C are denoted α, β, γ, respectively.

By considering the normals to the planes OAB and OAC, or otherwise, show that

$$\cos\alpha = \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})}{|\mathbf{a}\times\mathbf{b}||\mathbf{a}\times\mathbf{c}|}.$$

Using identities (1) and (2), prove that

$$\cos\delta(B,C) = \cos\delta(A,B)\cos\delta(A,C) + \sin\delta(A,B)\sin\delta(A,C)\cos\alpha$$

and

$$\frac{\sin\alpha}{\sin\delta(B,C)} = \frac{\sin\beta}{\sin\delta(A,C)} = \frac{\sin\gamma}{\sin\delta(A,B)}.$$

For an equilateral spherical triangle show that $\alpha > \pi/3$.

Paper 1, Section II

6B Vectors and Matrices

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the matrix equation $A\mathbf{x} = \mathbf{c}$ is 0, 1 or infinity, where A is a real 3×3 matrix and $\mathbf{c} \in \mathbb{R}^3$. State conditions on A and $\mathbf{c}$ that distinguish between these possibilities, and state the relationship that holds between any two solutions when there are infinitely many.

Consider the case

$$A = \begin{pmatrix} a & a & b \\ b & a & a \\ a & b & a \end{pmatrix} \quad \text{and } \mathbf{c} = \begin{pmatrix} 1 \\ c \\ 1 \end{pmatrix} .$$

Use row and column operations to find and factorize the determinant of A.

Find the kernel and image of the linear map represented by A for all values of a and b. Find the general solution to $A\mathbf{x} = \mathbf{c}$ for all values of a, b and c for which a solution exists.

Paper 1, Section II

7A Vectors and Matrices

Let A be an $n \times n$ Hermitian matrix. Show that all the eigenvalues of A are real.

Suppose now that A has n distinct eigenvalues.

(a) Show that the eigenvectors of A are orthogonal.

(b) Define the *characteristic polynomial* $P_A(t)$ of A. Let

$$P_A(t) = \sum_{r=0}^{n} a_r t^r \,.$$

Prove the matrix identity

$$\sum_{r=0}^{n} a_r A^r = 0 \,.$$

(c) What is the range of possible values of

$$\frac{\mathbf{x}^\dagger A \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}}$$

for non-zero vectors $\mathbf{x} \in \mathbb{C}^n$? Justify your answer.

(d) For any (not necessarily symmetric) real 2×2 matrix B with real eigenvalues, let $\lambda_{\max}(B)$ denote its maximum eigenvalue. Is it possible to find a constant C such that

$$\frac{\mathbf{x}^\dagger B \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} \leqslant C\, \lambda_{\max}(B)$$

for all non-zero vectors $\mathbf{x} \in \mathbb{R}^2$ and all such matrices B? Justify your answer.

Paper 1, Section II

8A Vectors and Matrices

(a) Explain what is meant by saying that a 2×2 real transformation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves the scalar product with respect to the Euclidean metric $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{R}^2$.

Derive a description of all such matrices that uses a single real parameter together with choices of sign (± 1). Show that these matrices form a group.

(b) Explain what is meant by saying that a 2×2 real transformation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves the scalar product with respect to the Minkowski metric $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{R}^2$.

Consider now the set of such matrices with $a > 0$. Derive a description of all matrices in this set that uses a single real parameter together with choices of sign (± 1). Show that these matrices form a group.

(c) What is the intersection of these two groups?