

1/I/1B Vectors and Matrices

State de Moivre's Theorem. By evaluating

$$\sum_{r=1}^n e^{ir\theta},$$

or otherwise, show that

$$\sum_{r=1}^n \cos(r\theta) = \frac{\cos(n\theta) - \cos((n+1)\theta)}{2(1 - \cos\theta)} - \frac{1}{2}.$$

Hence show that

$$\sum_{r=1}^n \cos\left(\frac{2p\pi r}{n+1}\right) = -1,$$

where p is an integer in the range $1 \leq p \leq n$.

1/I/2A Vectors and Matrices

Let U be an $n \times n$ unitary matrix ($U^\dagger U = UU^\dagger = I$). Suppose that A and B are $n \times n$ Hermitian matrices such that $U = A + iB$.

Show that

- (i) A and B commute,
- (ii) $A^2 + B^2 = I$.

Find A and B in terms of U and U^\dagger , and hence show that A and B are uniquely determined for a given U .

1/II/5B **Vectors and Matrices**

(a) Use suffix notation to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} .$$

Hence, or otherwise, expand

(i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$,

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

(b) Write down the equation of the line that passes through the point \mathbf{a} and is parallel to the unit vector $\hat{\mathbf{t}}$.

The lines L_1 and L_2 in three dimensions pass through \mathbf{a}_1 and \mathbf{a}_2 respectively and are parallel to the unit vectors $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ respectively. Show that a necessary condition for L_1 and L_2 to intersect is

$$(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\hat{\mathbf{t}}_1 \times \hat{\mathbf{t}}_2) = 0 .$$

Why is this condition not sufficient?

In the case in which L_1 and L_2 are non-parallel and non-intersecting, find an expression for the shortest distance between them.

1/II/6A **Vectors and Matrices**

A real 3×3 matrix A with elements A_{ij} is said to be *upper triangular* if $A_{ij} = 0$ whenever $i > j$. Prove that if A and B are upper triangular 3×3 real matrices then so is the matrix product AB .

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that $A^3 + A^2 - A = I$. Write A^{-1} as a linear combination of A^2 , A and I and hence compute A^{-1} explicitly.

For all integers n (including negative integers), prove that there exist coefficients α_n , β_n and γ_n such that

$$A^n = \alpha_n A^2 + \beta_n A + \gamma_n I.$$

For all integers n (including negative integers), show that

$$(A^n)_{11} = 1, \quad (A^n)_{22} = (-1)^n, \quad \text{and} \quad (A^n)_{23} = n(-1)^{n-1}.$$

Hence derive a set of 3 simultaneous equations for $\{\alpha_n, \beta_n, \gamma_n\}$ and find their solution.

1/II/7C Vectors and Matrices

Prove that any n orthonormal vectors in \mathbb{R}^n form a basis for \mathbb{R}^n .

Let A be a real symmetric $n \times n$ matrix with n orthonormal eigenvectors \mathbf{e}_i and corresponding eigenvalues λ_i . Obtain coefficients a_i such that

$$\mathbf{x} = \sum_i a_i \mathbf{e}_i$$

is a solution to the equation

$$A\mathbf{x} - \mu\mathbf{x} = \mathbf{f},$$

where \mathbf{f} is a given vector and μ is a given scalar that is not an eigenvalue of A .

How would your answer differ if $\mu = \lambda_1$?

Find a_i and hence \mathbf{x} when

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

in the cases (i) $\mu = 2$ and (ii) $\mu = 1$.

1/II/8C Vectors and Matrices

Prove that the eigenvalues of a Hermitian matrix are real and that eigenvectors corresponding to distinct eigenvalues are orthogonal (i.e. $\mathbf{e}_i^* \cdot \mathbf{e}_j = 0$).

Let A be a real 3×3 non-zero antisymmetric matrix. Show that iA is Hermitian. Hence show that there exists a (complex) eigenvector \mathbf{e}_1 such $A\mathbf{e}_1 = \lambda\mathbf{e}_1$, where λ is imaginary.

Show further that there exist real vectors \mathbf{u} and \mathbf{v} and a real number θ such that

$$A\mathbf{u} = \theta\mathbf{v} \quad \text{and} \quad A\mathbf{v} = -\theta\mathbf{u}.$$

Show also that A has a real eigenvector \mathbf{e}_3 such that $A\mathbf{e}_3 = 0$.

Let $R = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$. By considering the action of R on \mathbf{u} , \mathbf{v} and \mathbf{e}_3 , show that R is a rotation matrix.