

A1d

Vectors and Matrices: Example Sheet 4

Michaelmas 2009

A * denotes a question, or part of a question, that should not be done at the expense of questions later on the sheet. Starred questions are **not** necessarily harder than unstarred questions.

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1. A matrix A is said to be *upper triangular* if $A_{ij} = 0$ if $i > j$, i.e. if

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ 0 & A_{22} & \ddots & A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix}.$$

Show that the eigenvalues are $\lambda_i = A_{ii}$ ($i = 1, \dots, n$).

2. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be bases for \mathbb{R}^m and \mathbb{R}^n respectively, and let \mathcal{A} be a linear mapping from \mathbb{R}^m to \mathbb{R}^n . Explain how to represent \mathcal{A} by a matrix relative to the given bases.

- (a) Taking $m = 2$, $n = 3$ and \mathcal{A} as the linear mapping for which

$$\mathcal{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \mathcal{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix},$$

where components are with respect to the standard bases for \mathbb{R}^2 and \mathbb{R}^3 , find the matrix of \mathcal{A} with respect to the bases

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}; \quad \mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- (b) The mapping \mathcal{A} of \mathbb{R}^3 to itself is a reflection in the plane $x_1 \sin \theta = x_2 \cos \theta$. Find the matrix A of \mathcal{A} with respect to any basis of your choice, which should be specified.

3. The linear map $\mathcal{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.$$

Find the matrix B of the map \mathcal{A} relative to the basis

$$\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

and interpret the map geometrically. Hence show that, for each positive integer n ,

$$B^n - I = n(B - I),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence evaluate A^n . Verify that $\det(A^n) = (\det A)^n$.

- *4. Show that similar matrices have the same rank.

5. Show that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

has characteristic equation $(t - 2)^3 = 0$. Explain (without doing any further calculations) why A is not diagonalisable.

6. (a) Find a , b and c such that the matrix

$$\begin{pmatrix} 1/3 & 0 & a \\ 2/3 & 1/\sqrt{2} & b \\ 2/3 & -1/\sqrt{2} & c \end{pmatrix}$$

is orthogonal. Does this condition determine a , b and c uniquely?

- (b) Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Do you expect PAP^{-1} to be symmetric? Compute PAP^{-1} . Were you right?

- *7. (a) Show that the Cayley-Hamilton theorem is true for all 2×2 matrices.

- (b) Let

$$A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

Find the characteristic equation for A . Verify that $A^2 = 2A - I$. Is A diagonalisable?

Show by induction that A^n lies in the two-dimensional subspace (of the space of 2×2 real matrices) spanned by A and I , i.e. show that there exists real numbers α_n and β_n such that

$$A^n = \alpha_n A + \beta_n I.$$

Find a recurrence relation (i.e. a difference equation) for α_n and β_n , and hence find an explicit formula for A^n .

8. Determine the eigenvalues and eigenvectors of the symmetric matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Use an identity of the form $PAP^T = D$, where D is a diagonal matrix, to find A^{-1} .

- *9. Show that the eigenvalues of a unitary matrix have unit modulus. Show that if a unitary matrix has distinct eigenvalues then the eigenvectors are orthogonal.
10. A skew-Hermitian matrix, W , is one such that $W^\dagger = -W$. What can be said about the eigenvalues of a skew-Hermitian matrix? (*Hint: consider $H = iW$*)?

If S is real symmetric and T is real skew-symmetric, show that $T \pm iS$ is skew-Hermitian. State a property of the eigenvalues of $T + iS$ and hence, or otherwise, show that

$$\det(T + iS - I) \neq 0.$$

Show that the matrix

$$U = (I + T + iS)(I - T - iS)^{-1}$$

is unitary. For

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

show that the eigenvalues of U are $\pm(1 - i)/\sqrt{2}$.

- *11. This is a continuation of question 8 on Example Sheet 2.

As in question 8 on Example Sheet 2 consider the linear map $\mathcal{S} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b} \cdot \mathbf{x}) \mathbf{a} \quad (*)$$

where λ is a real scalar, \mathbf{a} and \mathbf{b} are fixed orthogonal unit vectors. Let $S(\lambda, \mathbf{a}, \mathbf{b})$ be the matrix with elements S_{ij} such that $x'_i = S_{ij}x_j$. Give diagrams illustrating the shears

$$S_1 = S(\lambda, \mathbf{i}, \mathbf{j}), \quad S_2 = S(\lambda, \mathbf{j}, -\mathbf{i}).$$

Show that S_1 and S_2 are related by a similarity transformation

$$S_2 = R^{-1}S_1R, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let \mathcal{S} be the map defined by (*) but from \mathbb{R}^3 to \mathbb{R}^3 , and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along the three perpendicular axes. Find the matrix \mathbf{S} in each of the cases

$$(i) \quad \mathbf{a} = \mathbf{i}, \quad \mathbf{b} = \mathbf{j}, \quad (ii) \quad \mathbf{a} = \mathbf{j}, \quad \mathbf{b} = \mathbf{k}, \quad (iii) \quad \mathbf{a} = \mathbf{k}, \quad \mathbf{b} = \mathbf{i},$$

and interpret the corresponding simple shears. Show that any matrix of the form

$$\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{pmatrix}$$

can be displayed (not necessarily uniquely) as the product of matrices of simple shears.

*12. Diagonalize the quadratic form

$$\mathcal{F} = (a \cos^2 \theta + b \sin^2 \theta)x^2 + 2(a - b)(\sin \theta \cos \theta)xy + (a \sin^2 \theta + b \cos^2 \theta)y^2.$$

13. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

$$A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Hence sketch the surfaces

$$5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1.$$

14. Let Σ be the surface in \mathbb{R}^3 given by

$$2x^2 + 2xy + 4yz + z^2 = 1.$$

By writing this equation as

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 1,$$

with \mathbf{A} a real symmetric matrix, show that there is an orthonormal basis such that, if we use coordinates (u, v, w) with respect to this new basis, Σ takes the form

$$\lambda u^2 + \mu v^2 + \nu w^2 = 1.$$

Find λ , μ and ν and hence find the minimum distance between the origin and Σ . *Hint: it is **not** necessary to find the basis explicitly.*

15. (i) Explain what is meant by saying that a 2×2 real matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

preserves the scalar product on \mathbb{R}^2 with respect to

$$(a) \text{ the Eulerian metric, } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or } (b) \text{ the Minkowski metric, } J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Using a single real parameter together with a choice of sign (± 1), give and justify a description of all matrices, \mathbf{A} , that preserve the scalar product with respect to the Euclidean metric. Show that these matrices form a group.

(iii) Using a single real parameter together with a choice of sign (± 1), give and justify a description of all matrices, \mathbf{A} with $a > 0$, that preserve the scalar product with respect to the Minkowski metric. Show that these matrices form a group.

(iv) What is the intersection of the above two groups?

Revision Questions

16. Show that a rotation about the z axis through an angle θ corresponds to the matrix

$$R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write down a real eigenvector of R and give the corresponding eigenvalue. In the case of a matrix corresponding to a general rotation, how can one find the axis of rotation?

A rotation through 45° about the x -axis is followed by a similar one about the z -axis. Show that the rotation corresponding to their combined effect has its axis inclined at equal angles

$$\cos^{-1} \frac{1}{\sqrt{(5 - 2\sqrt{2})}}$$

to the x and z axes.

17. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue so is λ^* . Hence argue that the eigenvalues of a 3×3 real orthogonal matrix Q must be a selection from

$$+1, \quad -1 \quad \text{and} \quad e^{i\alpha} \ \& \ e^{-i\alpha}.$$

Verify that $\det Q = \pm 1$. What is the effect of Q on vectors orthogonal to an eigenvector with eigenvalue ± 1 ?

- *18. *This is another way of proving $\det AB = \det A \det B$. You may wish to stick to the case $n = 3$.*

If $1 \leq r, s \leq n$, $r \neq s$ and λ is real, let $E(\lambda, r, s)$ be an $n \times n$ matrix with (i, j) entry $\delta_{ij} + \lambda \delta_{ir} \delta_{js}$. If $1 \leq r \leq n$ and μ is real, let $F(\mu, r)$ be an $n \times n$ matrix with (i, j) entry $\delta_{ij} + (\mu - 1) \delta_{ir} \delta_{jr}$.

- (i) Give a simple geometric interpretation of the linear maps from \mathbb{R}^n to \mathbb{R}^n associated with $E(\lambda, r, s)$ and $F(\mu, r)$.
 - (ii) Give a simple account of the effect of pre-multiplying an $n \times m$ matrix by $E(\lambda, r, s)$ and by $F(\mu, r)$. What is the effect of post-multiplying an $m \times n$ matrix?
 - (iii) If A is an $n \times n$ matrix, find $\det(E(\lambda, r, s)A)$ and $\det(F(\mu, r)A)$ in terms of $\det A$.
 - (iv) Show that every $n \times n$ matrix is the product of matrices of the form $E(\lambda, r, s)$ and $F(\mu, r)$ and a diagonal matrix with entries 0 or 1.
 - (v) Use (iii) and (iv) to show that, if A and B are $n \times n$ matrices, then $\det A \det B = \det AB$.
- *19. *The object of this exercise is to show why finding eigenvalues of a large matrix is not just a matter of finding a large fast computer.*

Consider the $n \times n$ complex matrix $A = \{A_{ij}\}$ given by

$$\begin{aligned} A_{j,j+1} &= 1 && \text{for } 1 \leq j \leq n-1 \\ A_{n1} &= \kappa^n \\ A_{ij} &= 0 && \text{otherwise,} \end{aligned}$$

where $\kappa \in \mathbb{C}$ is non-zero. Thus, when $n = 2$ and $n = 3$, we get the matrices

$$\begin{pmatrix} 0 & 1 \\ \kappa^2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \kappa^3 & 0 & 0 \end{pmatrix}.$$

- (i) Find the eigenvalues and associated eigenvectors of A for $n = 2$ and $n = 3$.
- (ii) By guessing and then verifying your answers, or otherwise, find the eigenvalues and associated eigenvectors of A for all $n \geq 2$.
- (iii) Suppose that your computer works to 15 decimal places and that $n = 100$. You decide to find the eigenvalues of A in the cases $\kappa = 2^{-1}$ and $\kappa = 3^{-1}$. Explain why at least one (and more probably) both attempts will deliver answers which bear no relation to the true answers.

The number of operations needed to calculate a $n \times n$ determinant using the Laplace expansion formulae.

Let f_n be the number of operations needed to calculate a $n \times n$ determinant using the Laplace expansion formulae. Since each $n \times n$ determinant requires us to calculate n smaller $(n-1) \times (n-1)$ determinants, plus perform n multiplications and $(n-1)$ additions,

$$f_n = nf_{n-1} + 2n - 1.$$

Hence

$$\frac{f_n + 2}{n!} = \frac{(f_{n-1} + 2)}{(n-1)!} + \frac{1}{n!},$$

and so by recursion

$$\frac{f_n + 2}{n!} = \frac{(f_2 + 2)}{2!} + \sum_{r=3}^n \frac{1}{r!}$$

But $f_2 = 3$, and so

$$\begin{aligned} f_n &= n! \sum_{r=0}^n \frac{1}{r!} - 2 \\ &= n!e - 2 - n! \sum_{r=n+1}^{\infty} \frac{1}{r!} \\ &\rightarrow n!e - 2 \end{aligned} \qquad \text{as } n \rightarrow \infty.$$