A1a **Vectors and Matrices: Example Sheet 1** Michaelmas 2008

A $*$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** *necessarily harder than unstarred questions.*

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1. Use complex numbers to show that the medians of a triangle are concurrent.

   Hint: represent the vertices of the triangle by complex numbers z_1, z_2 and z_3 (or 0, z_1 and z_2 if you prefer), then write down equations for two of the medians and find their intersection.

2. Let S be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in S$ then
$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.$$
   Obtain the same result geometrically by considering the line containing the centre of the circle and the point 3.

3. Show that
$$\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta,$$
   and hence that
$$\left(1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5 = 0.$$

4. Given $|z| = 1$ and $\arg z = \theta$, write in modulus argument form
$$\text{(i)}\quad 1 + z, \qquad \text{(ii)}\quad 1 - z.$$
   Show that w given by
$$w = \left(\frac{1 - z}{1 + z}\right)^{\frac{1}{2}}$$
   describes for varying θ two fixed straight lines in the complex plane.

5$*$ Express
$$I = \frac{z^5 - 1}{z - 1}$$
as a polynomial in z. By considering the complex fifth root of unity ω, obtain the four factors of I linear in z. Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I, show that
$$4\cos\frac{\pi}{5}\,\sin\frac{\pi}{10} = 1\ .$$

6. Explain why the equation $\sin z = 2$ has infinitely many solutions.

7. Let $z, a, b \in \mathbb{C}$ $(a \neq b)$ correspond to the points P, A, B of the Argand plane. Let C_λ be the locus of P defined by
$$PA/PB = \lambda$$
   where λ is a fixed real positive constant. Show that C_λ is a circle, unless $\lambda = 1$, and find its centre and radius. What if $\lambda = 1$?

8∗ Let C_λ be the circle of question 7 for the case $a = -b = p,\ p \in \mathbb{R}$. Show, for each fixed $\mu \in \mathbb{R}$, that the curve

$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$

is a circle passing through A and B with its centre on the perpendicular bisector of AB.

Show that the circles C_λ and S_μ are orthogonal for all λ, μ.

9. Show by vector methods that the altitudes of a triangle are concurrent.

Hint: let the altitudes AD, BE of $\triangle ABC$ meet at H, and show that CH is perpendicular to AB.

10. Given that vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy

$$\mathbf{x} + \mathbf{y}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{a}\ ,$$

for fixed vector $\mathbf{a}$, show that

$$(\mathbf{x} \cdot \mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2}\ .$$

Deduce using the Schwarz inequality (or otherwise) that

$$|\mathbf{x}|(1 + |\mathbf{y}|^2) \geqslant |\mathbf{a}| \geqslant |\mathbf{x}|\ .$$

Explain the circumstances under which either of the inequalities can be replaced by equalities, and describe the relation between $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ in these circumstances.

11. (a) In $\triangle ABC$, let $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ be denoted by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}\,, \qquad (*)$$

and hence obtain the sine rule for $\triangle ABC$.

(b) Given that $|\mathbf{u} \times \mathbf{v}| \neq 0$, show that $(*)$ implies

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}\,.$$

12. (a) Using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, deduce that

(i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$,

(ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

Relate the case $\mathbf{c} = \mathbf{a}$, $\mathbf{d} = \mathbf{b}$ of (i) to a well-known trigonometric identity.
Evaluate $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ in two distinct ways and use the result to display explicitly a linear dependence relation amongst the four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.

(b) Given $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, show that

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.$$

13. The set X contains the six vectors

$$(1, 1, 0, 0),\ (1, 0, 1, 0),\ (1, 0, 0, 1),\ (0, 1, 1, 0),\ (0, 1, 0, 1),\ (0, 0, 1, 1).$$

Find two different subsets Y of X whose members are linearly independent, each of which yields a linearly dependent subset of X whenever any element $\in X, \notin Y$, is adjoined to Y.

14. Let V be the set of all vectors $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ $(n \geqslant 4)$ such that

$$x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-3\,.$$

Find a basis for V.