

A \* denotes a question, or part of a question, that should not be done at the expense of questions later on the sheet. Starred questions are **not** necessarily harder than unstarred questions.

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1. A matrix  $A$  is said to be *upper triangular* if  $A_{ij} = 0$  if  $i > j$ , i.e. if

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ 0 & A_{22} & \ddots & A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix}.$$

Show that the eigenvalues are  $\lambda_i = A_{ii}$  ( $i = 1, \dots, n$ ).

2. Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$  and  $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$  be bases for  $\mathbb{R}^m$  and  $\mathbb{R}^n$  respectively, and let  $\mathcal{A}$  be a linear mapping from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Explain how to represent  $\mathcal{A}$  by a matrix relative to the given bases.

- (a) Taking  $m = 2$ ,  $n = 3$  and  $\mathcal{A}$  as the linear mapping for which

$$\mathcal{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \mathcal{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix},$$

where components are with respect to the standard bases for  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , find the matrix of  $\mathcal{A}$  with respect to the bases

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}; \quad \mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- (b) The mapping  $\mathcal{A}$  of  $\mathbb{R}^3$  to itself is a reflection in the plane  $x_1 \sin \theta = x_2 \cos \theta$ . Find the matrix  $A$  of  $\mathcal{A}$  with respect to any basis of your choice, which should be specified.

3. The linear map  $\mathcal{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.$$

Find the matrix  $B$  of the map  $\mathcal{A}$  relative to the basis

$$\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

and interpret the map geometrically. Hence show that, for each positive integer  $n$ ,

$$B^n - I = n(B - I),$$

where  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Hence evaluate  $A^n$ . Verify that  $\det(A^n) = (\det A)^n$ .

- \*4. Show that similar matrices have the same rank.

5. Show that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

has characteristic equation  $(t - 2)^3 = 0$ . Explain (without doing any further calculations) why  $A$  is not diagonalisable.

6. (a) Find  $a$ ,  $b$  and  $c$  such that the matrix

$$\begin{pmatrix} 1/3 & 0 & a \\ 2/3 & 1/\sqrt{2} & b \\ 2/3 & -1/\sqrt{2} & c \end{pmatrix}$$

is orthogonal. Does this condition determine  $a$ ,  $b$  and  $c$  uniquely?

- (b) Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Do you expect  $PAP^{-1}$  to be symmetric? Compute  $PAP^{-1}$ . Were you right?

- \*7. (a) Show that the Cayley-Hamilton theorem is true for all  $2 \times 2$  matrices.

- (b) Let

$$A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

Find the characteristic equation for  $A$ . Verify that  $A^2 = 2A - I$ . Is  $A$  diagonalisable?

Show by induction that  $A^n$  lies in the two-dimensional subspace (of the space of  $2 \times 2$  real matrices) spanned by  $A$  and  $I$ , i.e. show that there exists real numbers  $\alpha_n$  and  $\beta_n$  such that

$$A^n = \alpha_n A + \beta_n I.$$

Find a recurrence relation (i.e. a difference equation) for  $\alpha_n$  and  $\beta_n$ , and hence find an explicit formula for  $A^n$ .

8. Determine the eigenvalues and eigenvectors of the symmetric matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Use an identity of the form  $PAP^T = D$ , where  $D$  is a diagonal matrix, to find  $A^{-1}$ .

9. Show that the eigenvalues of a unitary matrix have unit modulus. Show that if a unitary matrix has distinct eigenvalues then the eigenvectors are orthogonal.
10. A skew-Hermitian matrix,  $W$ , is one such that  $W^\dagger = -W$ . What can be said about the eigenvalues of a skew-Hermitian matrix? (*Hint: consider  $H = iW$* )

If  $S$  is real symmetric and  $T$  is real skew-symmetric, show that  $T \pm iS$  is skew-Hermitian. State a property of the eigenvalues of  $T + iS$  and hence, or otherwise, show that

$$\det(T + iS - I) \neq 0.$$

Show that the matrix

$$U = (I + T + iS)(I - T - iS)^{-1}$$

is unitary. For

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

show that the eigenvalues of  $U$  are  $\pm(1 - i)/\sqrt{2}$ .

- \*11. This is a continuation of question 8 on Example Sheet 2.

As in question 8 on Example Sheet 2 consider the linear map  $S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b} \cdot \mathbf{x}) \mathbf{a} \quad (*)$$

where  $\lambda$  is a real scalar,  $\mathbf{a}$  and  $\mathbf{b}$  are fixed orthogonal unit vectors. Let  $S(\lambda, \mathbf{a}, \mathbf{b})$  be the matrix with elements  $S_{ij}$  such that  $x'_i = S_{ij}x_j$ . Give diagrams illustrating the shears

$$S_1 = S(\lambda, \mathbf{i}, \mathbf{j}), \quad S_2 = S(\lambda, \mathbf{j}, -\mathbf{i}).$$

Show that  $S_1$  and  $S_2$  are related by a similarity transformation

$$S_2 = R^{-1}S_1R, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let  $\mathcal{S}$  be the map defined by (\*) but from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , and let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be unit vectors along the three perpendicular axes. Find the matrix  $S$  in each of the cases

$$(i) \quad \mathbf{a} = \mathbf{i}, \quad \mathbf{b} = \mathbf{j}, \quad (ii) \quad \mathbf{a} = \mathbf{j}, \quad \mathbf{b} = \mathbf{k}, \quad (iii) \quad \mathbf{a} = \mathbf{k}, \quad \mathbf{b} = \mathbf{i},$$

and interpret the corresponding simple shears. Show that any matrix of the form

$$\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{pmatrix}$$

can be displayed (not necessarily uniquely) as the product of matrices of simple shears.

\*12. Diagonalize the quadratic form

$$\mathcal{F} = (a \cos^2 \theta + b \sin^2 \theta)x^2 + 2(a - b)(\sin \theta \cos \theta)xy + (a \sin^2 \theta + b \cos^2 \theta)y^2.$$

13. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

$$A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Hence sketch the surfaces

$$5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1.$$

14. Let  $\Sigma$  be the surface in  $\mathbb{R}^3$  given by

$$2x^2 + 2xy + 4yz + z^2 = 1.$$

By writing this equation as

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 1,$$

with  $\mathbf{A}$  a real symmetric matrix, show that there is an orthonormal basis such that, if we use coordinates  $(u, v, w)$  with respect to this new basis,  $\Sigma$  takes the form

$$\lambda u^2 + \mu v^2 + \nu w^2 = 1.$$

Find  $\lambda$ ,  $\mu$  and  $\nu$  and hence find the minimum distance between the origin and  $\Sigma$ . *Hint: it is **not** necessary to find the basis explicitly.*

### For Boxing Day and/or Revision

15. Show that a rotation about the  $z$  axis through an angle  $\theta$  corresponds to the matrix

$$R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write down a real eigenvector of  $R$  and give the corresponding eigenvalue. In the case of a matrix corresponding to a general rotation, how can one find the axis of rotation?

A rotation through  $45^\circ$  about the  $x$ -axis is followed by a similar one about the  $z$ -axis. Show that the rotation corresponding to their combined effect has its axis inclined at equal angles

$$\cos^{-1} \frac{1}{\sqrt{(5 - 2\sqrt{2})}}$$

to the  $x$  and  $z$  axes.

16. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if  $\lambda$  is an eigenvalue so is  $\lambda^*$ . Hence argue that the eigenvalues of a  $3 \times 3$  real orthogonal matrix  $Q$  must be a selection from

$$+1, \quad -1 \quad \text{and} \quad e^{i\alpha} \ \& \ e^{-i\alpha}.$$

Verify that  $\det Q = \pm 1$ . What is the effect of  $Q$  on vectors orthogonal to an eigenvector with eigenvalue  $\pm 1$ ?

- \*17. *This is another way of proving  $\det AB = \det A \det B$ . You may wish to stick to the case  $n = 3$ .*

If  $1 \leq r, s \leq n$ ,  $r \neq s$  and  $\lambda$  is real, let  $E(\lambda, r, s)$  be an  $n \times n$  matrix with  $(i, j)$  entry  $\delta_{ij} + \lambda \delta_{ir} \delta_{js}$ . If  $1 \leq r \leq n$  and  $\mu$  is real, let  $F(\mu, r)$  be an  $n \times n$  matrix with  $(i, j)$  entry  $\delta_{ij} + (\mu - 1) \delta_{ir} \delta_{jr}$ .

- (i) Give a simple geometric interpretation of the linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  associated with  $E(\lambda, r, s)$  and  $F(\mu, r)$ .
  - (ii) Give a simple account of the effect of pre-multiplying an  $n \times m$  matrix by  $E(\lambda, r, s)$  and by  $F(\mu, r)$ . What is the effect of post-multiplying an  $m \times n$  matrix?
  - (iii) If  $A$  is an  $n \times n$  matrix, find  $\det(E(\lambda, r, s)A)$  and  $\det(F(\mu, r)A)$  in terms of  $\det A$ .
  - (iv) Show that every  $n \times n$  matrix is the product of matrices of the form  $E(\lambda, r, s)$  and  $F(\mu, r)$  and a diagonal matrix with entries 0 or 1.
  - (v) Use (iii) and (iv) to show that, if  $A$  and  $B$  are  $n \times n$  matrices, then  $\det A \det B = \det AB$ .
- \*18. *The object of this exercise is to show why finding eigenvalues of a large matrix is not just a matter of finding a large fast computer.*

Consider the  $n \times n$  complex matrix  $A = \{A_{ij}\}$  given by

$$\begin{aligned} A_{j,j+1} &= 1 & \text{for } 1 \leq j \leq n-1 \\ A_{n1} &= \kappa^n \\ A_{ij} &= 0 & \text{otherwise,} \end{aligned}$$

where  $\kappa \in \mathbb{C}$  is non-zero. Thus, when  $n = 2$  and  $n = 3$ , we get the matrices

$$\begin{pmatrix} 0 & 1 \\ \kappa^2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \kappa^3 & 0 & 0 \end{pmatrix}.$$

- (i) Find the eigenvalues and associated eigenvectors of  $A$  for  $n = 2$  and  $n = 3$ .
- (ii) By guessing and then verifying your answers, or otherwise, find the eigenvalues and associated eigenvectors of  $A$  for all  $n \geq 2$ .
- (iii) Suppose that your computer works to 15 decimal places and that  $n = 100$ . You decide to find the eigenvalues of  $A$  in the cases  $\kappa = 2^{-1}$  and  $\kappa = 3^{-1}$ . Explain why at least one (and more probably) both attempts will deliver answers which bear no relation to the true answers.