

A1c

Vectors and Matrices: Example Sheet 3

Michaelmas 2008

$A *$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** necessarily harder than unstarred questions.

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1. Suppose that A and B are both Hermitian matrices. Show that $AB + BA$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.

2. Given that A is the real matrix

$$\begin{pmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{pmatrix},$$

show with the aid of row operations that

$$\det A = (a - b)(b - c)(c - a)(ab + bc + ca).$$

[Recall that the value of the determinant is unchanged if a linear combination of any two rows is added to the third row.]

3. Show that

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = x^3 + y^3 + z^3 - 3xyz \equiv \Delta.$$

Show, by row operations, that

$$x + y + z, \quad x + \omega y + \omega^2 z, \quad x + \omega^2 y + \omega z$$

are factors of Δ , where ω is a complex cube root of unity. Show, by considering the coefficients of x^3 , that Δ is equal to the product of the three indicated factors.

4. If A is a $(2n + 1) \times (2n + 1)$ antisymmetric matrix ($n \in \mathbb{N}$), calculate $\det A$.
5. Let D be the $n \times n$ matrix which has the entry p , $p \neq 1$, at each place on the main diagonal and unity in every other position. Show that $\det D = (p + n - 1)(p - 1)^{n-1}$.
6. Identify the cofactors Δ_{ij} of a_{ij} in the matrix

$$A = \{a_{ij}\} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}.$$

Verify the identity $a_{ij}\Delta_{ik} = \delta_{jk} \det A$, and hence construct the matrix A^{-1} .

Use your result to solve the equations

$$\begin{aligned} x + y + z &= 1, \\ x + 2y + 3z &= -5, \\ 3x - 2y + 2z &= 4. \end{aligned}$$

Verify that your answers for (x, y, z) do indeed satisfy the equations.

7. For each real value of t , determine whether or not there exist solutions to the simultaneous equations

$$\begin{aligned} x + y + z &= t \\ tx + 2z &= 3 \\ 3x + ty + 5z &= 7, \end{aligned}$$

exhibiting the most general form of such solutions when they exist.

- *8. Let A be a real 3×3 matrix, and let \mathbf{d} be a 3 component column vector. Explain briefly how the general solution of the matrix equation $A\mathbf{x} = \mathbf{d}$, where \mathbf{x} is a 3 component column vector, depends on the kernel and image of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.

Consider the case

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^2 & b^2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find the kernel and image of the corresponding map, noting the different possibilities according to different values of a and b .

For which values of a and b do the equations $A\mathbf{x} = \mathbf{d}$ have (i) a unique solution, (ii) more than one solution, (iii) no solution? For each pair (a, b) satisfying (ii), give the solutions as the sum of a fixed solution and the general solution of the corresponding homogeneous equations.

9. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

10. (a) Find a 3×3 real matrix with eigenvalues $1, i, -i$. *Hint:* think geometrically.
 (b) Construct a 3×3 non-zero real matrix which has all three eigenvalues zero.
11. (a) Let A be a square matrix such that $A^m = 0$ for some integer m . Show that every eigenvalue of A is zero.
 (b) Let A be a real 2×2 matrix which has non-zero non-real eigenvalues. Show that the non-diagonal elements of A are non-zero, but that the diagonal elements may be zero.
12. Let Q be a $(2n + 1) \times (2n + 1)$ orthogonal matrix ($n \in \mathbb{N}$) with $\det Q = 1$. Show that Q has a unit eigenvalue. Give a geometric interpretation of your result for 3×3 matrices.
- *13. Suppose that A is an $n \times n$ square matrix and that A^{-1} exists. Show that if A has characteristic equation $a_0 + a_1 t + \dots + a_n t^n = 0$, then A^{-1} has characteristic equation

$$(-1)^n \det(A^{-1})(a_n + a_{n-1}t + \dots + a_0 t^n) = 0.$$

Hints. Take $n = 3$ in this question if you wish, but treat the general case if you can. It should be clear that λ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1} , but this result says more than this (about multiplicities of eigenvalues). You should use properties of the determinant to solve this problem, for example, $\det(A) \det(B) = \det(AB)$. You should also state explicitly why we do not need to worry about zero eigenvalues.

14. For each of the three matrices below,
- compute their eigenvalues (as often happens in exercises and seldom in real life each eigenvalue is a small integer);
 - for each real eigenvalue λ compute the dimension of the eigenspace $\{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \lambda\mathbf{x}\}$;
 - determine whether or not the matrix is diagonalizable as a map of \mathbb{R}^3 into itself.

$$\begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}, \quad \begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}.$$

The number of operations needed to calculate a $n \times n$ determinant using the Laplace expansion formulae.

Let f_n be the number of operations needed to calculate a $n \times n$ determinant using the Laplace expansion formulae. Since each $n \times n$ determinant requires us to calculate n smaller $(n-1) \times (n-1)$ determinants, plus perform n multiplications and $(n-1)$ additions,

$$f_n = nf_{n-1} + 2n - 1.$$

Hence

$$\frac{f_n + 2}{n!} = \frac{(f_{n-1} + 2)}{(n-1)!} + \frac{1}{n!},$$

and so by recursion

$$\frac{f_n + 2}{n!} = \frac{(f_2 + 2)}{2!} + \sum_{r=3}^n \frac{1}{r!}$$

But $f_2 = 3$, and so

$$\begin{aligned} f_n &= n! \sum_{r=0}^n \frac{1}{r!} - 2 \\ &= n!e - 2 - n! \sum_{r=n+1}^{\infty} \frac{1}{r!} \\ &\rightarrow n!e - 2 \qquad \text{as } n \rightarrow \infty. \end{aligned}$$