Lent 2010 **VECTOR CALCULUS - EXAMPLES 1** **Gareth Taylor**

A star means "can save until later" and not necessarily "is harder".

1. The curve given parametrically by $(a\cos^3 t, a\sin^3 t)$ with $0 \leqslant t \leqslant 2\pi$ is called an *astroid.* Sketch it, and find its length.

2. The curve defined by $y^2 = x^3$ is called *Neile's parabola.* Sketch the segment of Neile's parabola with $0 \leqslant x \leqslant 4$, and find the length of this segment.

*3. In $\mathbb{R}^2$ a path is defined in polar coordinates by $r = f(\theta)$, $\alpha \leqslant \theta \leqslant \beta$, with a function $f \in C^1[\alpha, \beta]$. Show that the length of the path is

$$L = \int_\alpha^\beta \sqrt{\big(f(\theta)\big)^2 + \big(f'(\theta)\big)^2}\,\mathrm{d}\theta.$$

Sketch the paths $r = a\theta$ and $r = a(1 + \cos\theta)$, where for both $a > 0$ and $0 \leqslant \theta \leqslant 2\pi$. Calculate their lengths.

4. A circular helix is given by $\mathbf{x} = (a\cos t, a\sin t, ct)$. Calculate the tangent $\mathbf{t}$, principal normal $\mathbf{n}$, curvature κ, binormal $\mathbf{b}$, and torsion τ. Sketch the helix for $a, c > 0$, showing $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ at some point on the helix.

*5. (a) Explain why the tangent, principal normal and binormal form an orthonormal system.

Show that the torsion can be written as $\tau = \dfrac{1}{\kappa^2}\left[\mathbf{t}, \dfrac{\partial \mathbf{t}}{\partial s}, \dfrac{\partial^2 \mathbf{t}}{\partial s^2}\right]$, where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the scalar triple product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. Verify this identity for the helix in question 4.

(b) For the curve $\mathbf{x}(t) = (x(t), y(t))$, assuming that $x(t)$ and $y(t)$ are suitably differentiable, show that $\kappa = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

6. Find the minimum and maximum curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Comment on the case when $a = b$.

If you have done question 5, feel free to quote the formula you derived for κ.

7. Evaluate explicitly each of the line integrals

$$\text{(a)} \int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z), \qquad \text{(b)} \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z), \qquad \text{(c)} \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)$$

along (i) the straight line path joining the origin to $x = y = z = 1$, and (ii) the parabolic path given parametrically by $x = t, y = t, z = t^2$ with $0 \leqslant t \leqslant 1$.

For which of these integrals do the two paths give the same results, and why?

8. The force fields $\mathbf{F}$ and $\mathbf{G}$ are defined by $\mathbf{F} = (-y, x, z)$ and $\mathbf{G} = (y, x, z)$ respectively. Calculate the line integrals, $\int \mathbf{F}\cdot\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}\cdot\mathrm{d}\mathbf{x}$, that represent the work done in moving a particle from $(1, 0, 0)$ to $(-1, 0, \pi)$ along (i) the helix $\mathbf{x} = (\cos t, \sin t, t)$, and (ii) the straight line joining the points.

If, for one of the forces, your answers to (i) and (ii) are the same, then explain why.

9. Sketch the curve $x = a\sin 2t$, $y = b\cos t$. Find the area of either loop.

10. The closed curve C in the (x, y) plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area enclosed by C is A. By calculating the integrals explicitly, show that

$$\int_C (x^2 y \, \mathrm{d}x + xy^2 \, \mathrm{d}y) = \int_A (y^2 - x^2) \, \mathrm{d}A = \frac{104}{105} a^4,$$

where C is described anticlockwise.

11. Use the substitution $x = r\cos\theta$, $y = \frac{1}{2} r \sin\theta$ to evaluate

$$\int_A \frac{x^2}{x^2 + 4y^2} \, \mathrm{d}A,$$

where A is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

12. The region A is bounded by the line segments $x = 0, 0 \leqslant y \leqslant 1$; $y = 0, 0 \leqslant x \leqslant 1$; $y = 1, 0 \leqslant x \leqslant \frac{3}{4}$; and by an arc of the parabola $y^2 = 4(1 - x)$. Consider a mapping into the (x, y) plane from the (u, v) plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch A and also the two regions in the (u, v) plane which are mapped into it. Hence calculate

$$\int_A \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}.$$

*13. Show, without changing the order of integration, that

$$\int_0^1 \left[\int_0^1 \frac{x - y}{(x + y)^3} \, \mathrm{d}y \right] \mathrm{d}x = \frac{1}{2}, \qquad \text{and} \qquad \int_0^1 \left[\int_0^1 \frac{x - y}{(x + y)^3} \, \mathrm{d}x \right] \mathrm{d}y = -\frac{1}{2}.$$

Comment on these results.

14. For the tetrahedron V with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, evaluate the integral

$$\int_V x \, \mathrm{d}V.$$

Hence find the centre of volume $\frac{1}{V} \int_V \mathbf{x} \, \mathrm{d}V$.

15. A solid cone is bounded by the surface $\theta = \alpha$ (in spherical polar coordinates) and the surface $z = a$. Its mass density is $\rho_0 \cos\theta$. Show that its mass is $\frac{2\pi}{3} \rho_0 a^3 (\sec\alpha - 1)$.

16. **[Tripos, 2005/III/12]**

Express the integral

$$I = \int_0^\infty \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^x \mathrm{d}z \; x \, e^{-Ax/y - Bxy - Cyz}$$

in terms of the new variables $\alpha = x/y$, $\beta = xy$, $\gamma = yz$. Hence show that

$$I = \frac{1}{2A(A + B)(A + B + C)}.$$

Assume that A, B and C are positive.

Please send any corrections or comments to me at glt1000@cam.ac.uk

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VECTOR CALCULUS - EXAMPLES 2

Gareth Taylor

1. (a) Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Using suffix notation, show that

$$\boldsymbol{\nabla}\cdot(\psi\mathbf{v}) = (\mathbf{v}\cdot\boldsymbol{\nabla})\psi + \psi\boldsymbol{\nabla}\cdot\mathbf{v} \quad \text{and} \quad \boldsymbol{\nabla}\times(\psi\mathbf{v}) = \boldsymbol{\nabla}\psi\times\mathbf{v} + \psi\boldsymbol{\nabla}\times\mathbf{v}.$$

(b) Evaluate the divergence and curl of the following:

$$r\mathbf{x}, \qquad \mathbf{a}(\mathbf{x}\cdot\mathbf{b}), \qquad \mathbf{a}\times\mathbf{x}, \qquad \frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3},$$

where $r = |\mathbf{x}|$, and $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.

(c) Use suffix notation and the summation convention to show that $\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}$, where $\mathbf{a}$ is a uniform vector field, and that $\boldsymbol{\nabla}r^n = nr^{n-2}\mathbf{x}$.

Obtain the same results using spherical polar coordinates.

In spherical polars, for a function f of r and θ only, $\boldsymbol{\nabla}f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be vector fields. Show using suffix notation that

(i) $\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}$

(ii) $\operatorname{curl}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\operatorname{div}\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - \mathbf{v}\operatorname{div}\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}$

(iii) $\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\times\operatorname{curl}\mathbf{v} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + \mathbf{v}\times\operatorname{curl}\mathbf{u} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u}$

Deduce from (iii) that $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \operatorname{grad}(\frac{1}{2}u^2) - \mathbf{u}\times\operatorname{curl}\mathbf{u}$.

3. (a) A vector field $\mathbf{B}(\mathbf{x})$ is parallel to the normals of a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that $\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = 0$.

(b) The vector fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are everywhere parallel and are both solenoidal, i.e. $\boldsymbol{\nabla}\cdot\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Show that $\mathbf{B}\cdot\boldsymbol{\nabla}(v/B) = 0$, where $v = |\mathbf{v}|$ and $B = |\mathbf{B}| \neq 0$.

4. Obtain the equation of the plane which is tangent to the surface $z = 3x^2y\sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1,0,0)$ and North to be $(0,1,0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

5. (a) Let $\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2)$ be vector fields.

Compute explicitly the line integrals $\int \mathbf{F}\cdot d\mathbf{x}$ and $\int \mathbf{G}\cdot d\mathbf{x}$ from $(0,0,0)$ to $(1,1,1)$ along (i) the straight line joining the points, and (ii) the path $\mathbf{x}(t) = (t, t^2, t^2)$.

Show that only one of $\mathbf{F}$ and $\mathbf{G}$ is a conservative field and find a scalar potential for this one. Comment on the answers to your integrals in the light of this.

(b) Let $\mathbf{H} = \big(3x^2\tan z - y^2e^{-xy^2}\sin y, (\cos y - 2xy\sin y)e^{-xy^2}, x^3\sec^2 z\big)$ be a vector field. Show that $\mathbf{H}\cdot d\mathbf{x}$ is an exact differential and find the most general function that has $\mathbf{H}\cdot d\mathbf{x}$ as its differential.

Hence calculate the line integral $\int_{P_1}^{P_2}\mathbf{H}\cdot d\mathbf{x}$ from the point $P_1 = (0,0,0)$ to the point $P_2 = (1, \pi/2, \pi/4)$.

(c) Let $\mathbf{u} = \big((x\cos y + \cos y - y\sin y)e^x, (-x\sin y - \sin y - y\cos y)e^x\big)$. Show that $\mathbf{u}$ is solenoidal and find a vector potential for it in the form $\psi(x,y)\mathbf{e}_z$.

6. A curve C is given parametrically in Cartesian coordinates by

$$\mathbf{x}(t) = \big(\cos(\sin nt)\cos t, \cos(\sin nt)\sin t, \sin(\sin nt)\big), \quad 0 \leqslant t \leqslant 2\pi,$$

where n is some fixed integer.

Using spherical polar coordinates, sketch and describe C. Show that $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{x} = 2\pi$, where $\mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ and C is traversed in the direction of increasing t.

Show also that $\mathbf{F}$ is the gradient of a scalar. Comment on your results.

*7. Consider

$$\mathbf{A}(\mathbf{x}) = -\int_0^1 \mathbf{x}\times\mathbf{B}(\mathbf{x}t)t\ \mathrm{d}t.$$

Show that $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$ if $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ everywhere.

8. A fluid flow has the velocity vector $\mathbf{v} = (0, 0, z + a)$ in Cartesian coordinates, where a is a constant. Calculate the volume flux of fluid flowing across the open hemispherical surface $r = a$, $z \geqslant 0$, and also the volume flowing across the disc $r \leqslant a$, $z = 0$. Verify the divergence theorem holds. *Volume flux of fluid* $= \int_S \mathbf{v}\cdot\mathrm{d}\mathbf{S}$.

9. **[Tripos, 2002/III/4]**

State the divergence theorem. Consider the integral $I = \int_S r^n\mathbf{r}\cdot\mathrm{d}\mathbf{S}$, where $n > 0$ and S is the sphere of radius R centred at the origin. Evaluate I directly, and by means of the divergence theorem.

10. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$, and let S be the *open* surface

$$x^2 + y^2 = 1 - z, \quad 0 \leqslant z \leqslant 1.$$

Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\displaystyle\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$.

Verify your result by calculating the integral directly.

You should find that $\mathrm{d}\mathbf{S} = (2\rho\cos\varphi, 2\rho\sin\varphi, 1)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi$.

11. Verify Stokes' theorem for the open hemispherical surface $r = 1$, $z \geqslant 0$, and the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, z)$.

12. By applying the divergence theorem to the vector field $\mathbf{k}\times\mathbf{B}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{B}(\mathbf{x})$ is a vector field, show that

$$\int_V \boldsymbol{\nabla}\times\mathbf{B}\ \mathrm{d}V = -\int_A \mathbf{B}\times\mathrm{d}\mathbf{A},$$

where the surface A encloses the volume V.

Verify this result when A is the sphere $|\mathbf{x}| = R$ and $\mathbf{B} = (z, 0, 0)$ in Cartesian coordinates.

13. By applying Stokes' theorem to the vector field $\mathbf{k}\times\mathbf{B}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{B}(\mathbf{x})$ is a vector field, show that

$$\oint_C \mathrm{d}\mathbf{x}\times\mathbf{B} = \int_A (\mathrm{d}\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{B},$$

where the curve C bounds the open surface A.

Verify this result when C is the unit square in the (x, y) plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$, and $\mathbf{B} = \mathbf{x}$.

14. **[Tripos, 2005/III/10]**

    Write down Stokes' theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

    Consider the bounded surface S defined by $z = x^2 + y^2$, $\frac{1}{4} \leqslant z \leqslant 1$. Sketch the surface and calculate the surface element $\mathrm{d}\mathbf{S}$. For the vector field $\mathbf{B} = (-y^3, x^3, z^3)$, calculate $I = \int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S}$ directly.

    Show using Stokes' theorem that I may be rewritten as a line integral and verify this yields the same result.

*15. **[Tripos, 2001/III/11]**

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \boldsymbol{\nabla}\Omega \; \mathrm{d}V = \int_S \Omega \; \mathrm{d}\mathbf{S}. \quad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x, y, z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \leqslant r \leqslant \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$, where a is a constant.

*16. **[Tripos, 2002/III/11]**

The first part of the question was to prove question 2(ii) above, so I have omitted it here.

S is an open orientable surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ is any continuously differentiable vector field such that $\mathbf{n}\cdot\mathbf{v} = 0$ on S. Let $\mathbf{m}$ be a continuously differentiable unit vector field which coincides with $\mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S (\delta_{ij} - n_i n_j) \frac{\partial v_i}{\partial x_j} \; \mathrm{d}S = \oint_C \mathbf{u} \cdot \mathbf{v} \; \mathrm{d}s,$$

where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}\,ds = ds \times \mathbf{n}$.

Verify this result by taking $\mathbf{v} = \mathbf{r}$ and S to be the disc $|\mathbf{r}| \leqslant R$ in the $z = 0$ plane.

Please send any corrections or comments to me at glt1000@cam.ac.uk

VECTOR CALCULUS - EXAMPLES 3 **Gareth Taylor**

1. Show that the unit basis vectors of cylindrical polar coordinates satisfy

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r,$$

with all other derivatives of the three basis vectors being zero.

Given that the vector differential operator $\boldsymbol{\nabla}$ in cylindrical polars is

$$\boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z},$$

obtain expressions for $\boldsymbol{\nabla} \cdot \mathbf{A}$ and $\boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A} = A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_z \mathbf{e}_z$.

2. The vector field $\mathbf{B}(\mathbf{x})$ is given in cylindrical polars by

$$\mathbf{B}(\mathbf{x}) = \frac{1}{r}\mathbf{e}_\theta.$$

Using the formula derived in the previous question, show that $\boldsymbol{\nabla} \times \mathbf{B} = 0$ when $r \neq 0$. Calculate $\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{x}$ with C the circle $r = 1$, $0 \leqslant \theta \leqslant 2\pi$, $z = 0$. Why does Stokes' theorem not apply?

3. If $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ in the volume V, and $\mathbf{J} \cdot \mathbf{n} = 0$ on the surface S which encloses V, show that

$$\int_V \mathbf{J} \, \mathrm{d}V = 0.$$

Consider $\frac{\partial}{\partial x_j}(x_i J_j)$.

4. Let the surface S enclose the volume V, and let $\mathbf{P}(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$ be two solenoidal vectors (i.e., $\boldsymbol{\nabla} \cdot \mathbf{P} = \boldsymbol{\nabla} \cdot \mathbf{Q} = 0$). Show that

$$\int_V (\mathbf{Q} \cdot \nabla^2 \mathbf{P} - \mathbf{P} \cdot \nabla^2 \mathbf{Q}) \, \mathrm{d}V = \int_S \big(\mathbf{Q} \times (\boldsymbol{\nabla} \times \mathbf{P}) - \mathbf{P} \times (\boldsymbol{\nabla} \times \mathbf{Q})\big) \cdot \mathrm{d}\mathbf{S}.$$

5. The scalar function φ depends only on the radial distance $r = |\mathbf{x}|$ in $\mathbb{R}^3$. Use Cartesian coordinates and the chain rule to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\frac{\mathbf{x}}{r}, \quad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\varphi'(r).$$

What are the corresponding results when working in $\mathbb{R}^2$ rather than $\mathbb{R}^3$?

Find the solution of $\nabla^2\varphi = 1$ in the region $r \leqslant 1$ in $\mathbb{R}^3$ which is not singular at the origin and satisfies $\varphi(1) = 1$.

6. Show that the radially symmetric solutions of Laplace's equation in two dimensions have the form $\varphi = \alpha + \beta \log r$, where α and β are constants.

What is the corresponding solution in three dimensions?

7. Find all solutions of Laplace's equation, $\nabla^2 f = 0$, in two dimensions that can be written in the separable form $f(r,\theta) = R(r)\Phi(\theta)$, where r and θ are plane polar coordinates.

Hence solve, for $r < a$, the following boundary value problem, assuming that $f(r,\theta)$ satisfies a reasonable physical condition at $r = 0$.

$$\nabla^2 f = 0, \ f(a,\theta) = \sin\theta$$

Find also the solution for $r > a$ that satisfies $f(r,\theta) \to 0$ as $r \to \infty$.

8. A spherical shell has density given by

$$\rho(r) = \begin{cases} 0 & \text{for } 0 < r < a \\ \rho_0 r/a & \text{for } a < r < b \\ 0 & \text{for } b < r < \infty \end{cases}$$

Find the gravitational field everywhere by three different methods, namely

(a) direct solution of Poisson's equation,

(b) Gauss's flux theorem,

*(c) the integral form of the general solution of Poisson, $\varphi(\mathbf{x}_0) = -\dfrac{1}{4\pi}\displaystyle\int_V \frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}V$.

You should assume that the potential is a function only of r, is not singular at the origin and that the potential and its first derivative are continuous at $r = a$ and $r = b$. What is the justification for these assumptions?

*9. **[Tripos, 2004/III/11]**

Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0,0,0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2},0,0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(-\frac{1}{4},0,0)$.

The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1.

Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

10. The surface S encloses a volume in which the scalar field φ satisfies the Klein-Gordon equation $\nabla^2\varphi = m^2\varphi$, where m is a real non-zero constant. Prove that φ is uniquely determined if either φ or $\partial\varphi/\partial n$ is given on S. *Recall that $\partial\varphi/\partial n = \mathbf{n}\cdot\boldsymbol{\nabla}\varphi$.*

11. Show that the solution to Laplace's equation with boundary condition

$$g\,\frac{\partial\varphi}{\partial n} + \varphi = f$$

is unique if $g(\mathbf{x}) \geqslant 0$ on the boundary.

Find a non-zero (and so non-unique) solution of Laplace's equation on $r \leqslant 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $r = 1$.

12. Let $u(\mathbf{x})$ be the unique solution of Laplace's equation in the volume V, subject to the boundary condition that u is equal to a given function $f(\mathbf{x})$ on the surface S which encloses V. Let v be any function with continuous first partial derivatives in V which vanishes on S. Show that

$$\int_V \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v\;\mathrm{d}V = 0.$$

Let w be a function with continuous first partial derivatives in V which satisfies $w = f$ on S. Use the above result with $v = w - u$ to deduce that

$$\int_V |\boldsymbol{\nabla}w|^2\;\mathrm{d}V \geqslant \int_V |\boldsymbol{\nabla}u|^2\;\mathrm{d}V,$$

i.e. the solution of the Laplace problem minimises $\int_V |\boldsymbol{\nabla}w|^2\,\mathrm{d}V$.

13. The scalar field φ is harmonic in a volume V bounded by a closed surface S. Given that V does not contain the origin ($r = 0$), show that

$$\int_S \left(\varphi \boldsymbol{\nabla} \left(\frac{1}{r} \right) - \left(\frac{1}{r} \right) \boldsymbol{\nabla} \varphi \right) \cdot \mathrm{d}\mathbf{S} = 0.$$

Now let V be the volume given by $\varepsilon \leqslant r \leqslant a$ and let S_1 be the surface $r = a$. Given that $\varphi(\mathbf{x})$ is harmonic for $r \leqslant a$, use this result, in the limit $\varepsilon \to 0$, to show that

$$\varphi(0) = \frac{1}{4\pi a^2} \int_{S_1} \varphi(\mathbf{x}) \ \mathrm{d}S.$$

Deduce that if φ is harmonic in a general volume V, then it attains its maximum and minimum values on S. *Note: 'harmonic' means 'satisfies Laplace's equation'.*

*14. If $\nabla^2 \varphi = \rho$ in a volume V enclosed by S and $\mathbf{x}_0$ is a point within V, show that

$$4\pi\varphi(\mathbf{x}_0) = -\int_V \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|} \ \mathrm{d}V + \int_S \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \frac{\partial \varphi}{\partial n}(\mathbf{x}) - \varphi(\mathbf{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right) \right) \mathrm{d}S.$$

You will find the methods of the previous question useful.

*15. **[Tripos, 2005/III/9]**

The first part of the question was essentially question 12 above. So I have omitted it here, but bear that context in mind in what follows.

Consider the partial differential equation $\dfrac{\partial w}{\partial t} = \nabla^2 w$, for $w = w(t, \mathbf{x})$, with initial condition $w(0, \mathbf{x}) = w_0(\mathbf{x})$ in V, and boundary condition $w(t, \mathbf{x}) = f(\mathbf{x})$ on S for all $t \geqslant 0$.

Show that

$$\frac{\partial}{\partial t} \int_V |\boldsymbol{\nabla} w|^2 \ \mathrm{d}V \leqslant 0, \qquad (*)$$

with equality holding only when $w(t, \mathbf{x}) = u(\mathbf{x})$. *(Note: this is the u from question 12.)*

Show that $(*)$ remains true with the boundary condition

$$\frac{\partial w}{\partial t} + \alpha(\mathbf{x}) \frac{\partial w}{\partial n} = 0$$

on S, provided $\alpha(\mathbf{x}) \geqslant 0$.

Please send any corrections or comments to me at glt1000@cam.ac.uk

Lent 2010 VECTOR CALCULUS - EXAMPLES 4 Gareth Taylor

1. A physical entity is represented in each Cartesian frame by the array of numbers δ_{ij} $(i, j = 1, 2, 3)$, equal to 1 when $i = j$ and 0 when $i \neq j$. Show that this entity is a tensor, (i) by showing directly that it has the appropriate transformation property, and (ii) by applying the quotient theorem.

   What is meant by saying that δ_{ij} is an *isotropic* tensor?

2. If $\mathbf{u}(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second-rank tensor.

   If $\sigma(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij}/\partial x_j$ transforms as a vector.

3. Show that if tensor A_{ij} is symmetric (or antisymmetric) and transforms to A'_{ij} under rotation of the coordinate axes, then A'_{ij} is also symmetric (or antisymmetric).

4. Any 3×3 matrix A can be decomposed in the form $A\mathbf{x} = \alpha\mathbf{x} + \boldsymbol{\omega} \times \mathbf{x} + B\mathbf{x}$, where α is a scalar, $\boldsymbol{\omega}$ is a vector, and B is a traceless symmetric matrix. Verify this claim by finding α, ω_k and B_{ij} explicitly in terms of A_{ij}.

   Explain why α, $\boldsymbol{\omega}$, and B together provide a space of the correct dimension to parameterise an arbitrary 3×3 matrix.

   Check your calculations are correct by finding α, $\boldsymbol{\omega}$ and B for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$.

*5. **[Tripos, 2005/I/8 – Algebra & Geometry]**

   Given a non-zero vector v_i, any 3×3 symmetric matrix T_{ij} can be expressed as $T_{ij} = A\delta_{ij} + Bv_iv_j + (C_iv_j + C_jv_i) + D_{ij}$ for some numbers A and B, some vector C_i and a symmetric matrix D_{ij}, where $C_iv_i = 0$, $D_{ii} = 0$, $D_{ij}v_j = 0$, and the summation convention is implicit.

   Show that the above statement is true by finding A, B, C_i and D_{ij} explicitly in terms of T_{ij} and v_j, or otherwise. Explain why A, B, C_i and D_{ij} together provide a space of the correct dimension to parameterise an arbitrary symmetric 3×3 matrix T_{ij}.

6. The electrical conductivity tensor σ_{ij} has components

$$\sigma_{ij} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

   Find the directions along which (i) no current flows, and (ii) the current is largest.

7. A body has the symmetry that its shape is unchanged by rotations of π about three perpendicular axes which form a basis $\mathcal{B}$. Show that any second-rank tensor calculated for the body will be diagonal in $\mathcal{B}$, although the diagonal elements need not be equal.

   Find the inertia tensor of a cuboid of uniform density with sides of length $2a$, $2b$ and $2c$ about the centre of the cuboid. Hence show that the moment of inertia of the cuboid about one of its long diagonals is

$$\frac{M}{3R^2}(R^4 - a^4 - b^4 - c^4),$$

   where M is the mass of the cuboid, and R is the distance from the centre of the cuboid to a corner.

8. A second rank tensor is defined in terms of the position vector $\mathbf{x}$ by $T_{ij} = \delta_{ij} + \varepsilon_{ijk}x_k$. Calculate the following integrals, with S being the surface of the unit sphere.

$$\text{(i)} \quad \int_S x_i \, \mathrm{d}S, \qquad \text{(ii)} \quad \int_S T_{ij} \, \mathrm{d}S, \qquad \text{(iii)} \quad \int_S T_{ij}T_{jk} \, \mathrm{d}S.$$

9. Evaluate the following integrals over the whole of $\mathbb{R}^3$ for positive γ and $r^2 = x_p x_p$:

$$\text{(i)} \quad \int r^{-3}e^{-\gamma r^2} x_i x_j \, \mathrm{d}V, \qquad \text{(ii)} \quad \int r^{-5}e^{-\gamma r^2} x_i x_j x_k \, \mathrm{d}V.$$

10. For any second-rank tensor T_{ij}, prove using the transformation law that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \text{and} \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are the same in all bases.

If T_{ij} is a symmetric tensor, express these invariants in terms of the eigenvalues. Deduce that the cubic equation for the eigenvalues is

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

11. Given that the most general isotropic rank 4 tensor is $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$ for $\lambda, \mu, \nu \in \mathbb{R}$, show that

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

12. Given a non-zero vector v_i, the orthogonal projection tensor P_{ij} is defined by

$$P_{ij} = \delta_{ij} - \frac{v_i v_j}{v_k v_k}.$$

(i) Verify that P_{ij} satisfies (a) $P_{ij}v_j = 0$ and (b) $P_{ij}u_j = u_i$ for any vector u_i which is orthogonal to v_i.

(ii) Show that P_{ij} is unique: that is, if another tensor T_{ij} satisfies both (a) and (b), then $(P_{ij} - T_{ij})w_j = 0$ for any vector w_i.

(iii) For $A_{ij} = \varepsilon_{ijk}v_k$, show that $P_{ij}A_{jk}A_{km} = -v_k v_k P_{im}$.

*13. Three Cartesian frames of reference in $\mathbb{R}^3$ are such that the i^{th} axis of the first frame coincides with the $(i+n)^{\text{th}}$ axis (modulo 3) of the $(n+1)^{\text{th}}$ frame ($n = 0, 1, 2$). A physical entity has components

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the three frames, respectively. Show that this entity cannot be a tensor.

Show that the entity with respective components

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

could be a tensor, and on the assumption that it is, find its components in an arbitrary Cartesian frame whose axes are at angles θ_1, θ_2 and θ_3 to the 1-axis of the first frame, with $\cos\theta_1 = \lambda$, $\cos\theta_2 = \mu$ and $\cos\theta_3 = \nu$ (i.e., the axes have direction cosines λ, μ, ν with respect to the 1-axis of the first frame).

*14. The array D_{ikm} with 3^3 elements is not known to represent a tensor. If, for every symmetric tensor represented by a_{km}, $b_i = D_{ikm}a_{km}$ represents a vector, what can be said about the transformation properties under rotations of the coordinates axes of

$$(i)\ \ D_{ikm}, \qquad (ii)\ \ D_{ikm} + D_{imk}\ ?$$

15. A conductor positioned in a magnetic field $\mathbf{H}$ carries a steady current density $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{H}$, and the magnetic flux intensity $\mathbf{B} = \mu\mathbf{H}$ satisfies $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. The mechanical force per unit volume acting on the conductor can be written as $\mathbf{J} \times \mathbf{B}$. If the permeability μ is a constant, show that this force per unit volume can be written as $\partial s_{ik}/\partial x_k$ in terms of a tensor

$$s_{ik} = \mu(H_iH_k - \tfrac{1}{2}H_mH_m\delta_{ik}).$$

16. In linear elasticity, the symmetric second-rank stress tensor σ_{ij} is linear in the symmetric second-rank strain tensor e_{kl}. Show that in an isotropic material,

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

with two material constants λ and μ. *(You may quote the form of the general isotropic fourth-rank tensor.)* Solve the above equation to find an expression for e_{ij} in terms of σ_{kl}. Show that the eigenvectors of σ are parallel to the eigenvectors of e.

*17. For an ionized gas in a magnetic field $\mathbf{B}$ (note $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$) the pressure tensor p_{ij} (the negative of the stress tensor) takes the diagonal form

$$\begin{pmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{pmatrix}$$

in local axes with O_z parallel to $\mathbf{B}$. Here $p_\perp$ and $p_\parallel$ are scalar functions of position. Show that the divergence $\partial p_{ij}/\partial x_j$ of the pressure tensor takes the vector form

$$\boldsymbol{\nabla} p_\perp + (\mathbf{B} \cdot \boldsymbol{\nabla})\left(\frac{p_\parallel - p_\perp}{B^2}\right)\mathbf{B}.$$

*18. A vector field u_i has the following components in a particular system of Cartesian coordinates x_i:

$$u_1 = x_1x_2^2, \quad u_2 = x_2x_3^2, \quad u_3 = x_3x_1^2.$$

Express the tensor $\partial u_i/\partial x_k$ at the point $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ as a linear combination of $\varepsilon_{ijk}w_j$ (where w_j is a vector to be determined) and a symmetric tensor e_{ik}. Verify that the principal axes of e_{ik} are in the directions

$$\frac{1}{\sqrt{5}}(1, -2, 0), \quad \frac{1}{\sqrt{5}}(2, 1, 0) \quad \text{and} \quad (0, 0, 1).$$

∞. If you left any unstarred questions on any of these sheets, attempt them over the Easter break.

Please send any corrections or comments to me at glt1000@cam.ac.uk