A star means "can save until later" and not necessarily "is harder".

1. The curve given parametrically by $(a\cos^3 t, a\sin^3 t)$ with $0 \leqslant t \leqslant 2\pi$ is called an *astroid.* Sketch it, and find its length.

2. The curve defined by $y^2 = x^3$ is called *Neile's parabola.* Sketch the segment of Neile's parabola with $0 \leqslant x \leqslant 4$, and find the length of this segment.

*3. In $\mathbb{R}^2$ a path is defined in polar coordinates by $r = f(\theta)$, $\alpha \leqslant \theta \leqslant \beta$, with a function $f \in C^1[\alpha, \beta]$. Show that the length of the path is

$$L = \int_\alpha^\beta \sqrt{\big(f(\theta)\big)^2 + \big(f'(\theta)\big)^2}\,\mathrm{d}\theta.$$

Sketch the paths $r = a\theta$ and $r = a(1 + \cos\theta)$, where for both $a > 0$ and $0 \leqslant \theta \leqslant 2\pi$. Calculate their lengths.

4. Given a function $f(r)$ in two dimensions, use the chain rule to express its partial derivatives with respect to Cartesian coordinates (x, y) in terms of its partial derivatives with respect to polar coordinates (ρ, ϕ). From the relationship between the basis vectors in these coordinate systems, deduce that

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi.$$

5. In three dimensions, use suffix notation and the summation convention to show that

$$\text{(i)} \quad \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a} \quad \text{and} \quad \text{(ii)} \quad \boldsymbol{\nabla} r^n = n r^{n-2}\mathbf{x},$$

where $\mathbf{a}$ is any constant vector, and $r = |\mathbf{x}|$.

Obtain the same results using spherical polar coordinates.

In spherical polars, for a function f of r and θ only, $\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$.

6. Evaluate explicitly each of the line integrals

$$\text{(a)} \int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z), \quad \text{(b)} \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z), \quad \text{(c)} \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)$$

along (i) the straight line path joining the origin to $x = y = z = 1$, and (ii) the parabolic path given parametrically by $x = t, y = t, z = t^2$ with $0 \leqslant t \leqslant 1$.

For which of these integrals do the two paths give the same results, and why?

7. Obtain the equation of the plane which is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

8. Let $\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2)$ be vector fields.

Compute explicitly the line integrals $\int \mathbf{F}\cdot\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}\cdot\mathrm{d}\mathbf{x}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along (i) the straight line joining the points, and (ii) the path $\mathbf{x}(t) = (t, t^2, t^2)$.

Show that only one of $\mathbf{F}$ and $\mathbf{G}$ is a conservative field and find a scalar potential for this one. Comment on the answers to your integrals in the light of this.

9. A curve C is given parametrically in Cartesian coordinates by

$$\mathbf{x}(t) = \big(\cos(\sin nt)\cos t,\ \cos(\sin nt)\sin t,\ \sin(\sin nt)\big), \quad 0 \leqslant t \leqslant 2\pi,$$

where n is some fixed integer.

Using spherical polar coordinates, sketch and describe C. Show that $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{x} = 2\pi$, where $\mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ and C is traversed in the direction of increasing t.

Show also that $\mathbf{F}$ is the gradient of a scalar. Comment on your results.

10. Use the substitution $x = r\cos\theta$, $y = \frac{1}{2}r\sin\theta$ to evaluate

$$\int_A \frac{x^2}{x^2+4y^2}\,\mathrm{d}A,$$

where A is the region between the two ellipses $x^2+4y^2=1$, $x^2+4y^2=4$.

11. The closed curve C in the (x,y) plane consists of the arc of the parabola $y^2 = 4ax$ $(a>0)$ between the points $(a,\pm 2a)$ and the straight line joining $(a,\mp 2a)$. The area enclosed by C is A. By calculating the integrals explicitly, show that

$$\int_C (x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y) = \int_A (y^2-x^2)\,\mathrm{d}A = \frac{104}{105}a^4,$$

where C is described anticlockwise.

12. The region A is bounded by the line segments $\{x=0,\ 0\leqslant y\leqslant 1\}$, $\{y=0,\ 0\leqslant x\leqslant 1\}$, $\{y=1,\ 0\leqslant x\leqslant \frac{3}{4}\}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the (x,y) plane from the (u,v) plane defined by the transformation $x = u^2-v^2$, $y = 2uv$. Sketch A and also the two regions in the (u,v) plane which are mapped into it. Hence calculate

$$\int_A \frac{\mathrm{d}A}{(x^2+y^2)^{1/2}}.$$

*13. Without changing the order of integration, show that

$$\int_0^1\left[\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y\right]\mathrm{d}x = \frac{1}{2}, \qquad \text{and} \qquad \int_0^1\left[\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}x\right]\mathrm{d}y = -\frac{1}{2}.$$

Comment on these results.

14. Let T be the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Find the volume V of T, and hence find the centre of volume, given by

$$\frac{1}{V}\int_T \mathbf{x}\,\mathrm{d}V\ .$$

15. A solid cone is bounded by the surface $\theta = \alpha$ (in spherical polar coordinates) and the surface $z = a$. Its mass density is $\rho_0\cos\theta$. Show that its mass is $\frac{2\pi}{3}\rho_0 a^3(\sec\alpha - 1)$.

16. **[Tripos, 2005/III/12]**

Express the integral

$$I = \int_0^\infty \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^x \mathrm{d}z\ x\, e^{-Ax/y - Bxy - Cyz}$$

in terms of the new variables $\alpha = x/y$, $\beta = xy$, $\gamma = yz$. Hence show that

$$I = \frac{1}{2A(A+B)(A+B+C)}.$$

Assume that A, B and C are positive.

1. A circular helix is given by $\mathbf{x} = (a\cos t, a\sin t, ct)$. Calculate the tangent $\mathbf{t}$, principal normal $\mathbf{n}$, curvature κ, binormal $\mathbf{b}$, and torsion τ. Sketch the helix for $a, c > 0$, showing $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ at some point on the helix.

*2. Explain why the tangent, principal normal and binormal form an orthonormal system.

Show that the torsion can be written as

$$\tau = \frac{1}{\kappa^2}\left[\mathbf{t}, \frac{\partial \mathbf{t}}{\partial s}, \frac{\partial^2 \mathbf{t}}{\partial s^2}\right],$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the scalar triple product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

Verify this identity for the helix in question 1.

3. *(a) Show that a curve in the plane given by $\mathbf{x}(t) = (x(t), y(t))$ has curvature

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

(b) Find the minimum and maximum curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Comment on the case when $a = b$.

If you have done (a), feel free to quote the formula you derived for κ.

4. (a) Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Using suffix notation, show that

$$\boldsymbol{\nabla} \cdot (\psi \mathbf{v}) = (\boldsymbol{\nabla}\psi) \cdot \mathbf{v} + \psi \boldsymbol{\nabla} \cdot \mathbf{v} \quad \text{and} \quad \boldsymbol{\nabla} \times (\psi \mathbf{v}) = (\boldsymbol{\nabla}\psi) \times \mathbf{v} + \psi \boldsymbol{\nabla} \times \mathbf{v}.$$

(b) Evaluate the divergence and curl of the following:

$$r\mathbf{x}, \quad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \quad \mathbf{a} \times \mathbf{x}, \quad \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^3},$$

where $r = |\mathbf{x}|$, and $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.

5. Let $\mathbf{u}$ and $\mathbf{v}$ be vector fields. Show using suffix notation that

(i) $\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{u} \cdot (\boldsymbol{\nabla} \times \mathbf{v})$

(ii) $\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\boldsymbol{\nabla} \cdot \mathbf{v}) + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} - \mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{u}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v}$

(iii) $\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v} + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{u}) + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u}$

Deduce from (iii) that $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}u^2) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u})$.

6. Show that the vector field

$$\mathbf{H} = \left(3x^2 \tan z - y^2 e^{-xy^2} \sin y, (\cos y - 2xy \sin y)e^{-xy^2}, x^3 \sec^2 z\right)$$

is conservative, and find the most general scalar potential for $\mathbf{H}$. Hence calculate the line integral $\int_{P_1}^{P_2} \mathbf{H} \cdot \mathrm{d}\mathbf{x}$ from the point $P_1 = (0, 0, 0)$ to the point $P_2 = (1, \pi/2, \pi/4)$.

7. Show that the vector field

$$\mathbf{u} = e^x(x\cos y + \cos y - y\sin y)\mathbf{i} + e^x(-x\sin y - \sin y - y\cos y)\mathbf{j}$$

is irrotational and express it as the gradient of a scalar field ϕ. Show that $\mathbf{u}$ is also solenoidal and find a vector potential for it in the form $\psi\mathbf{k}$, for some function ψ.

8. (a) A vector field $\mathbf{B}(\mathbf{x})$ is parallel to the normals of a family of surfaces $f(\mathbf{x}) =$ constant. Show that $\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = 0$.

   (b) The vector fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are everywhere parallel and are both solenoidal. Show that $\mathbf{B} \cdot \boldsymbol{\nabla}(v/B) = 0$, where $v = |\mathbf{v}|$ and $B = |\mathbf{B}| \neq 0$.

   *(c) The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $|\mathbf{H}|^{-3}|\mathbf{H} \times (\mathbf{H} \cdot \boldsymbol{\nabla})\mathbf{H}|$.

*9. Consider

$$\mathbf{A}(\mathbf{x}) = -\int_0^1 \mathbf{x} \times \mathbf{B}(\mathbf{x}t)t \, \mathrm{d}t.$$

Show that $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ if $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ everywhere.

10. A fluid flow has the velocity vector $\mathbf{v} = (0, 0, z + a)$ in Cartesian coordinates, where a is a constant. Calculate the volume flux of fluid flowing across the open hemispherical surface $r = a$, $z \geqslant 0$, and also that flowing across the disc $r \leqslant a$, $z = 0$. Verify the divergence theorem holds. *[Volume flux of fluid* $= \int_S \mathbf{v} \cdot \mathrm{d}\mathbf{S}$*.]*

11. **[Tripos, 2002/III/4]**

State the divergence theorem. Consider the integral $I = \int_S r^n \mathbf{r} \cdot \mathrm{d}\mathbf{S}$, where $n > 0$ and S is the sphere of radius R centred at the origin. Evaluate I directly, and by means of the divergence theorem.

12. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$, and let S be the *open* surface

$$x^2 + y^2 = 1 - z, \quad 0 \leqslant z \leqslant 1.$$

Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\displaystyle\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

Verify your result by calculating the integral directly.

[You should find that $\mathrm{d}\mathbf{S} = (2\rho\cos\varphi, 2\rho\sin\varphi, 1)\rho \, \mathrm{d}\rho \, \mathrm{d}\varphi$*.]*

13. Verify Stokes' theorem for the open hemispherical surface $r = 1$, $z \geqslant 0$, and the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, z)$.

14. By applying the divergence theorem to the vector field $\mathbf{k} \times \mathbf{B}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{B}(\mathbf{x})$ is a vector field, show that

$$\int_V \boldsymbol{\nabla} \times \mathbf{B} \, \mathrm{d}V = -\int_A \mathbf{B} \times \mathrm{d}\mathbf{A},$$

where the surface A encloses the volume V.

Verify this result when A is the sphere $|\mathbf{x}| = R$ and $\mathbf{B} = (z, 0, 0)$ in Cartesian coordinates.

15. By applying Stokes' theorem to the vector field $\mathbf{k} \times \mathbf{B}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{B}(\mathbf{x})$ is a vector field, show that

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{B} = \int_A (\mathrm{d}\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{B},$$

where the curve C bounds the open surface A.

Verify this result when C is the unit square in the (x, y) plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$, and $\mathbf{B} = \mathbf{x}$.

16. **[Tripos, 2005/III/10]**

Write down Stokes' theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

Consider the bounded surface S defined by $z = x^2 + y^2$, $\frac{1}{4} \leqslant z \leqslant 1$. Sketch the surface and calculate the surface element $\mathrm{d}\mathbf{S}$. For the vector field $\mathbf{B} = (-y^3, x^3, z^3)$, calculate $I = \int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S}$ directly.

Show using Stokes' theorem that I may be rewritten as a line integral and verify this yields the same result.

*17. **[Tripos, 2001/III/11]**

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \boldsymbol{\nabla}\Omega \ \mathrm{d}V = \int_S \Omega \ \mathrm{d}\mathbf{S}. \quad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x, y, z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \leqslant r \leqslant \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$, where a is a constant.

*18. **[Tripos, 2002/III/11]**

The first part of the question was to prove question 5(ii) above, so I have omitted it here.

S is an open orientable surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ is any continuously differentiable vector field such that $\mathbf{n} \cdot \mathbf{v} = 0$ on S. Let $\mathbf{m}$ be a continuously differentiable unit vector field which coincides with $\mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S (\delta_{ij} - n_i n_j) \frac{\partial v_i}{\partial x_j} \ \mathrm{d}S = \oint_C \mathbf{u} \cdot \mathbf{v} \ \mathrm{d}s,$$

where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}\,ds = d\mathbf{s} \times \mathbf{n}$.

Verify this result by taking $\mathbf{v} = \mathbf{r}$ and S to be the disc $|\mathbf{r}| \leqslant R$ in the $z = 0$ plane.

1. (a) Write down the operator $\boldsymbol{\nabla}$ in Cartesian coordinates and in spherical polars, and calculate the gradient of $\psi = Ez = Ercos\theta$ in both coordinate systems. By considering the relationship between the basis vectors, check that your answers agree.

   (b) Calculate, in three ways, the curl of the vector field

$$\mathbf{A} \;=\; \tfrac{1}{2}B(-y\mathbf{e}_x + x\mathbf{e}_y) \;=\; \tfrac{1}{2}B\rho\mathbf{e}_\phi \;=\; \tfrac{1}{2}Br\sin\theta\mathbf{e}_\phi,$$

   by applying the standard formulas in Cartesian, cylindrical, and spherical coordinates.

2. Show that the unit basis vectors of cylindrical polar coordinates satisfy

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r,$$

   with all other derivatives of the three basis vectors being zero.

   Given that the vector differential operator $\boldsymbol{\nabla}$ in cylindrical polars is

$$\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z},$$

   obtain expressions for $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$.

3. The vector field $\mathbf{B}(\mathbf{x})$ is given in cylindrical polar coordinates (r, θ, z) by

$$\mathbf{B}(\mathbf{x}) = \frac{1}{r}\mathbf{e}_\theta.$$

   Using the formula derived in question 2, show that $\boldsymbol{\nabla}\times\mathbf{B} = 0$ when $r \neq 0$. Calculate $\oint_C \mathbf{B}\cdot \mathrm{d}\mathbf{x}$ with C the circle $r = 1$, $0 \leqslant \theta \leqslant 2\pi$, $z = 0$. Why does Stokes' theorem not apply?

4. Let $\mathbf{J}$ satisfy $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$ in the volume V and $\mathbf{J}\cdot\mathbf{n} = 0$ on the boundary ∂V. By considering $\frac{\partial}{\partial x_j}(x_i J_j)$, show that

$$\int_V \mathbf{J}\;\mathrm{d}V = 0.$$

*5. Let the surface S enclose the volume V, and let $\mathbf{P}(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$ be two solenoidal vectors (i.e., $\boldsymbol{\nabla}\cdot\mathbf{P} = \boldsymbol{\nabla}\cdot\mathbf{Q} = 0$). Show that

$$\int_V (\mathbf{Q}\cdot\nabla^2\mathbf{P} - \mathbf{P}\cdot\nabla^2\mathbf{Q})\;\mathrm{d}V = \int_S \big(\mathbf{Q}\times(\boldsymbol{\nabla}\times\mathbf{P}) - \mathbf{P}\times(\boldsymbol{\nabla}\times\mathbf{Q})\big)\cdot \mathrm{d}\mathbf{S}.$$

6. (a) The scalar function φ depends only on the radial distance $r = |\mathbf{x}|$ in $\mathbb{R}^3$. Use Cartesian coordinates and the chain rule to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\frac{\mathbf{x}}{r}, \quad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\varphi'(r).$$

   What are the corresponding results when working in $\mathbb{R}^2$ rather than $\mathbb{R}^3$?

   (b) Show that the radially symmetric solutions of Laplace's equation in two dimensions have the form $\varphi = \alpha + \beta\log r$, where α and β are constants.

   (c) Find the solution of $\nabla^2\varphi = 1$ in the region $r \leqslant 1$ in $\mathbb{R}^3$ which is not singular at the origin and satisfies $\varphi(1) = 1$.

7. Find all solutions of Laplace's equation, $\nabla^2 f = 0$, in two dimensions that can be written in the separable form $f(r,\theta) = R(r)\Phi(\theta)$, where r and θ are plane polar coordinates.

   Hence solve, for $r < a$, the following boundary value problem, assuming that $f(r,\theta)$ satisfies a reasonable physical condition at $r = 0$.

$$\nabla^2 f = 0, \;\; f(a,\theta) = \sin\theta$$

   Find also the solution for $r > a$ that satisfies $f(r,\theta) \to 0$ as $r \to \infty$.

8. A spherical shell has density given by

$$\rho(r) = \begin{cases} 0 & \text{for } 0 < r < a \\ \rho_0 r/a & \text{for } a < r < b \\ 0 & \text{for } b < r < \infty \end{cases}$$

   Find the gravitational field everywhere by three different methods, namely

   (a) direct solution of Poisson's equation,

   (b) Gauss's flux theorem,

   *(c) the integral form of the general solution of Poisson, $\varphi(\mathbf{x}_0) = -\dfrac{1}{4\pi}\displaystyle\int_V \frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}V$.

   You should assume that the potential is a function only of r, is not singular at the origin and that the potential and its first derivative are continuous at $r = a$ and $r = b$.

9. Show that Maxwell's equations for $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ imply that the charge density $\rho(\mathbf{x},t)$ and current density $\mathbf{j}(\mathbf{x},t)$ satisfy the conservation equation $\boldsymbol{\nabla}\cdot\mathbf{j} = -\partial\rho/\partial t$. Show also that if $\mathbf{j}$ is zero then

$$U = \tfrac{1}{2}\epsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2) \quad \text{and} \quad \mathbf{P} = \mu_0^{-1}\mathbf{E}\times\mathbf{B} \quad \text{satisfy} \quad \boldsymbol{\nabla}\cdot\mathbf{P} = -\partial U/\partial t\ .$$

*10. **[Tripos, 2004/III/11]**

   Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0,0,0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2},0,0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(-\frac{1}{4},0,0)$.

   The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1.

   Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

11. The surface S encloses a volume in which the scalar field φ satisfies the Klein-Gordon equation $\nabla^2\varphi = m^2\varphi$, where m is a real non-zero constant. Prove that φ is uniquely determined if either φ or $\partial\varphi/\partial n$ is given on S. *Recall that $\partial\varphi/\partial n = \mathbf{n}\cdot\boldsymbol{\nabla}\varphi$.*

12. Show that the solution to Laplace's equation in a volume V with boundary condition

$$g\frac{\partial\varphi}{\partial n} + \varphi = f \quad \text{on } \partial V$$

   is unique if $g(\mathbf{x}) \geqslant 0$ on ∂V.

   Find a non-zero (and so non-unique) solution of Laplace's equation on $r \leqslant 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $r = 1$.

13. The functions $u(\mathbf{x})$ and $v(\mathbf{x})$ on V satisfy $\nabla^2 u = 0$ on V and $v = 0$ on ∂V . Show that

$$\int_V \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \; \mathrm{d}V = 0.$$

Let w be a function on V which satisfies $w = u$ on ∂V. By considering $v = w - u$, show that

$$\int_V |\boldsymbol{\nabla} w|^2 \; \mathrm{d}V \geqslant \int_V |\boldsymbol{\nabla} u|^2 \; \mathrm{d}V,$$

i.e. the solution of the Laplace problem minimises $\int_V |\boldsymbol{\nabla} w|^2 \, \mathrm{d}V$.

14. The scalar field φ is harmonic in a volume V bounded by a closed surface S. Given that V does not contain the origin ($r = 0$), show that

$$\int_S \left(\varphi \boldsymbol{\nabla} \left(\frac{1}{r} \right) - \left(\frac{1}{r} \right) \boldsymbol{\nabla} \varphi \right) \cdot \mathrm{d}\mathbf{S} = 0.$$

Now let V be the volume given by $\varepsilon \leqslant r \leqslant a$ and let S_1 be the surface $r = a$. Given that $\varphi(\mathbf{x})$ is harmonic for $r \leqslant a$, use this result, in the limit $\varepsilon \to 0$, to show that

$$\varphi(0) = \frac{1}{4\pi a^2} \int_{S_1} \varphi(\mathbf{x}) \; \mathrm{d}S.$$

Deduce that if φ is harmonic in a general volume V, then it attains its maximum and minimum values on S. *Note: 'harmonic' means 'satisfies Laplace's equation'.*

*15. If $\nabla^2 \varphi = \rho$ in a volume V enclosed by S and $\mathbf{x}_0$ is a point within V, show that

$$4\pi\varphi(\mathbf{x}_0) = -\int_V \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|} \; \mathrm{d}V + \int_S \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \frac{\partial \varphi}{\partial n}(\mathbf{x}) - \varphi(\mathbf{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right) \right) \mathrm{d}S.$$

You will find the methods of the previous question useful.

*16. **[Tripos, 2005/III/9]**

The first part of the question was essentially question 13 above. So I have omitted it here, but bear that context in mind in what follows.

Consider the partial differential equation $\dfrac{\partial w}{\partial t} = \nabla^2 w$, for $w = w(t, \mathbf{x})$, with initial condition $w(0, \mathbf{x}) = w_0(\mathbf{x})$ in V, and boundary condition $w(t, \mathbf{x}) = f(\mathbf{x})$ on S for all $t \geqslant 0$.

Show that

$$\frac{\partial}{\partial t} \int_V |\boldsymbol{\nabla} w|^2 \; \mathrm{d}V \leqslant 0, \qquad (*)$$

with equality holding only when $w(t, \mathbf{x}) = u(\mathbf{x})$. *(Note: this is the u from question 13.)*

Show that $(*)$ remains true with the boundary condition

$$\frac{\partial w}{\partial t} + \alpha(\mathbf{x}) \frac{\partial w}{\partial n} = 0$$

on S, provided $\alpha(\mathbf{x}) \geqslant 0$.

1. A physical entity is represented in each Cartesian frame by the array of numbers δ_{ij} $(i, j = 1, 2, 3)$, equal to 1 when $i = j$ and 0 when $i \neq j$. Show that this entity is a tensor, (i) by showing directly that it has the appropriate transformation property, and (ii) by applying the quotient theorem.

   What is meant by saying that δ_{ij} is an *isotropic* tensor?

2. If $\mathbf{u}(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second-rank tensor.

   If $\sigma(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij}/\partial x_j$ transforms as a vector.

3. Show that if tensor A_{ij} is symmetric (or antisymmetric) and transforms to A'_{ij} under rotation of the coordinate axes, then A'_{ij} is also symmetric (or antisymmetric).

4. Any 3×3 matrix A can be decomposed in the form $A\mathbf{x} = \alpha\mathbf{x} + \boldsymbol{\omega} \times \mathbf{x} + B\mathbf{x}$, where α is a scalar, $\boldsymbol{\omega}$ is a vector, and B is a traceless symmetric matrix. Verify this claim by finding α, ω_k and B_{ij} explicitly in terms of A_{ij}.

   Explain why α, $\boldsymbol{\omega}$, and B together provide a space of the correct dimension to parameterise an arbitrary 3×3 matrix.

   Check your calculations are correct by finding α, $\boldsymbol{\omega}$ and B for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$.

*5. **[Tripos, 2005/I/8 – Algebra & Geometry]**

   Given a non-zero vector v_i, any 3×3 symmetric matrix T_{ij} can be expressed as $T_{ij} = A\delta_{ij} + Bv_iv_j + (C_iv_j + C_jv_i) + D_{ij}$ for some numbers A and B, some vector C_i and a symmetric matrix D_{ij}, where $C_iv_i = 0$, $D_{ii} = 0$, $D_{ij}v_j = 0$, and the summation convention is implicit.

   Show that the above statement is true by finding A, B, C_i and D_{ij} explicitly in terms of T_{ij} and v_j, or otherwise. Explain why A, B, C_i and D_{ij} together provide a space of the correct dimension to parameterise an arbitrary symmetric 3×3 matrix T_{ij}.

6. The electrical conductivity tensor σ_{ij} has components

   $$\sigma_{ij} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

   Find the directions along which (i) no current flows, and (ii) the current is largest.

7. A body has the symmetry that its shape is unchanged by rotations of π about three perpendicular axes which form a basis $\mathcal{B}$. Show that any second-rank tensor calculated for the body will be diagonal in $\mathcal{B}$, although the diagonal elements need not be equal.

   Find the inertia tensor of a cuboid of uniform density with sides of length $2a$, $2b$ and $2c$ about the centre of the cuboid. Hence show that the moment of inertia of the cuboid about one of its long diagonals is

   $$\frac{M}{3R^2}(R^4 - a^4 - b^4 - c^4),$$

   where M is the mass of the cuboid, and R is the distance from the centre of the cuboid to a corner.

8. A second rank tensor is defined in terms of the position vector $\mathbf{x}$ by $T_{ij} = \delta_{ij} + \varepsilon_{ijk} x_k$. Calculate the following integrals, with S being the surface of the unit sphere.

$$\text{(i)} \quad \int_S x_i \, \mathrm{d}S, \qquad \text{(ii)} \quad \int_S T_{ij} \, \mathrm{d}S, \qquad \text{(iii)} \quad \int_S T_{ij} T_{jk} \, \mathrm{d}S.$$

9. Evaluate the following integrals over the whole of $\mathbb{R}^3$ for positive γ and $r^2 = x_p x_p$:

$$\text{(i)} \quad \int r^{-3} e^{-\gamma r^2} x_i x_j \, \mathrm{d}V, \qquad \text{(ii)} \quad \int r^{-5} e^{-\gamma r^2} x_i x_j x_k \, \mathrm{d}V.$$

10. For any second-rank tensor T_{ij}, prove using the transformation law that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij} T_{ji}, \quad \text{and} \quad \gamma = T_{ij} T_{jk} T_{ki}$$

are the same in all bases.

If T_{ij} is a symmetric tensor, express these invariants in terms of the eigenvalues. Deduce that the cubic equation for the eigenvalues is

$$\lambda^3 - \alpha \lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

11. Given that the most general isotropic rank 4 tensor is $\lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{jk}$ for $\lambda, \mu, \nu \in \mathbb{R}$, show that

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}.$$

12. Given a non-zero vector v_i, the orthogonal projection tensor P_{ij} is defined by

$$P_{ij} = \delta_{ij} - \frac{v_i v_j}{v_k v_k}.$$

(i) Verify that P_{ij} satisfies (a) $P_{ij} v_j = 0$ and (b) $P_{ij} u_j = u_i$ for any vector u_i which is orthogonal to v_i.

(ii) Show that P_{ij} is unique: that is, if another tensor T_{ij} satisfies both (a) and (b), then $(P_{ij} - T_{ij}) w_j = 0$ for any vector w_i.

(iii) For $A_{ij} = \varepsilon_{ijk} v_k$, show that $P_{ij} A_{jk} A_{km} = -v_k v_k P_{im}$.

*13. Three Cartesian frames of reference in $\mathbb{R}^3$ are such that the i^{th} axis of the first frame coincides with the $(i+n)^{\text{th}}$ axis (modulo 3) of the $(n+1)^{\text{th}}$ frame ($n = 0, 1, 2$). A physical entity has components

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the three frames, respectively. Show that this entity cannot be a tensor.

Show that the entity with respective components

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

could be a tensor, and on the assumption that it is, find its components in an arbitrary Cartesian frame whose axes are at angles θ_1, θ_2 and θ_3 to the 1-axis of the first frame, with $\cos\theta_1 = \lambda$, $\cos\theta_2 = \mu$ and $\cos\theta_3 = \nu$ (i.e., the axes have direction cosines λ, μ, ν with respect to the 1-axis of the first frame).

*14. The array D_{ikm} with 3^3 elements is not known to represent a tensor. If, for every symmetric tensor represented by a_{km}, $b_i = D_{ikm}a_{km}$ represents a vector, what can be said about the transformation properties under rotations of the coordinates axes of

$$(i)\ \ D_{ikm}, \qquad (ii)\ \ D_{ikm} + D_{imk}\ ?$$

15. A conductor positioned in a magnetic field $\mathbf{H}$ carries a steady current density $\mathbf{J} = \nabla \times \mathbf{H}$, and the magnetic flux intensity $\mathbf{B} = \mu\mathbf{H}$ satisfies $\nabla \cdot \mathbf{B} = 0$. The mechanical force per unit volume acting on the conductor can be written as $\mathbf{J} \times \mathbf{B}$. If the permeability μ is a constant, show that this force per unit volume can be written as $\partial s_{ik}/\partial x_k$ in terms of a tensor

$$s_{ik} = \mu(H_i H_k - \tfrac{1}{2} H_m H_m \delta_{ik}).$$

16. In linear elasticity, the symmetric second-rank stress tensor σ_{ij} is linear in the symmetric second-rank strain tensor e_{kl}. Show that in an isotropic material,

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

with two material constants λ and μ. *(You may quote the form of the general isotropic fourth-rank tensor.)* Solve the above equation to find an expression for e_{ij} in terms of σ_{kl}. Show that the eigenvectors of σ are parallel to the eigenvectors of e.

*17. For an ionized gas in a magnetic field $\mathbf{B}$ (note $\nabla \cdot \mathbf{B} = 0$) the pressure tensor p_{ij} (the negative of the stress tensor) takes the diagonal form

$$\begin{pmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{pmatrix}$$

in local axes with O_z parallel to $\mathbf{B}$. Here $p_\perp$ and $p_\parallel$ are scalar functions of position. Show that the divergence $\partial p_{ij}/\partial x_j$ of the pressure tensor takes the vector form

$$\nabla p_\perp + (\mathbf{B} \cdot \nabla) \left(\frac{p_\parallel - p_\perp}{B^2} \right) \mathbf{B}.$$

*18. A vector field u_i has the following components in a particular system of Cartesian coordinates x_i:

$$u_1 = x_1 x_2^2, \quad u_2 = x_2 x_3^2, \quad u_3 = x_3 x_1^2.$$

Express the tensor $\partial u_i/\partial x_k$ at the point $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ as a linear combination of $\varepsilon_{ijk} w_j$ (where w_j is a vector to be determined) and a symmetric tensor e_{ik}. Verify that the principal axes of e_{ik} are in the directions

$$\frac{1}{\sqrt{5}}(1, -2, 0), \quad \frac{1}{\sqrt{5}}(2, 1, 0) \quad \text{and} \quad (0, 0, 1).$$

∞. If you left any unstarred questions on any of these sheets, attempt them over the Easter break.

Please send any corrections or comments to me at glt1000@cam.ac.uk