Paper 3, Section I

3C Vector Calculus

Cartesian coordinates x, y, z and spherical polar coordinates r, θ, ϕ are related by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \ .$$

Find scalars h_r, h_θ, h_ϕ and unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ such that

$$\mathrm{d}\mathbf{x} = h_r\mathbf{e}_r\,\mathrm{d}r + h_\theta\mathbf{e}_\theta\,\mathrm{d}\theta + h_\phi\mathbf{e}_\phi\,\mathrm{d}\phi \ .$$

Verify that the unit vectors are mutually orthogonal.

Hence calculate the area of the open surface defined by $\theta = \alpha$, $0 \leqslant r \leqslant R$, $0 \leqslant \phi \leqslant 2\pi$, where α and R are constants.

Paper 3, Section I

4C Vector Calculus

State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$, where $r = |\mathbf{x}|$.

Vector fields $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are given by $\mathbf{u} = r^\alpha\mathbf{x}$ and $\mathbf{v} = \mathbf{k}\times\mathbf{u}$, where α is a constant and $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$, and deduce that $\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$ and $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. When $\alpha = -3$, show that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and

$$\boldsymbol{\nabla}\times\mathbf{v} = \frac{3(\mathbf{k}\cdot\mathbf{x})\mathbf{x} - \mathbf{k}r^2}{r^5} \ .$$

Paper 3, Section II
9C Vector Calculus

Write down the most general isotropic tensors of rank 2 and 3. Use the tensor transformation law to show that they are, indeed, isotropic.

Let V be the sphere $0 \leqslant r \leqslant a$. Explain briefly why

$$T_{i_1 \dots i_n} = \int_V x_{i_1} \dots x_{i_n} \, \mathrm{d}V$$

is an isotropic tensor for any n. Hence show that

$$\int_V x_i x_j \, \mathrm{d}V = \alpha \delta_{ij}, \quad \int_V x_i x_j x_k \, \mathrm{d}V = 0 \quad \text{and} \int_V x_i x_j x_k x_l \, \mathrm{d}V = \beta(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

for some scalars α and β, which should be determined using suitable contractions of the indices or otherwise. Deduce the value of

$$\int_V \mathbf{x} \times (\mathbf{\Omega} \times \mathbf{x}) \, \mathrm{d}V \ ,$$

where $\mathbf{\Omega}$ is a constant vector.

[*You may assume that the most general isotropic tensor of rank 4 is*

$$\lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk} \ ,$$

where λ, μ and ν are scalars.]

Paper 3, Section II
10C Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{x})$ in a region V bounded by a piecewise smooth surface S with outward normal $\mathbf{n}$.

Show, by suitable choice of $\mathbf{u}$, that

$$\int_V \boldsymbol{\nabla} f \, \mathrm{d}V = \int_S f \, \mathrm{d}\mathbf{S} \qquad (*)$$

for a scalar field $f(\mathbf{x})$.

Let V be the paraboloidal region given by $z \geqslant 0$ and $x^2 + y^2 + cz \leqslant a^2$, where a and c are positive constants. Verify that $(*)$ holds for the scalar field $f = xz$.

Paper 3, Section II

11C Vector Calculus

The electric field $\mathbf{E}(\mathbf{x})$ due to a static charge distribution with density $\rho(\mathbf{x})$ satisfies

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \ , \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \ , \tag{1}$$

where $\phi(\mathbf{x})$ is the corresponding electrostatic potential and ε_0 is a constant.

(a) Show that the total charge Q contained within a closed surface S is given by Gauss' Law

$$Q = \varepsilon_0 \int_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} \ .$$

Assuming spherical symmetry, deduce the electric field and potential due to a point charge q at the origin i.e. for $\rho(\mathbf{x}) = q\,\delta(\mathbf{x})$.

(b) Let $\mathbf{E}_1$ and $\mathbf{E}_2$, with potentials ϕ_1 and ϕ_2 respectively, be the solutions to (1) arising from two different charge distributions with densities ρ_1 and ρ_2. Show that

$$\frac{1}{\varepsilon_0}\int_V \phi_1\rho_2 \,\mathrm{d}V + \int_{\partial V} \phi_1 \boldsymbol{\nabla}\phi_2\cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \phi_2\rho_1 \,\mathrm{d}V + \int_{\partial V} \phi_2 \boldsymbol{\nabla}\phi_1\cdot \mathrm{d}\mathbf{S} \tag{2}$$

for any region V with boundary ∂V, where $\mathrm{d}\mathbf{S}$ points out of V.

(c) Suppose that $\rho_1(\mathbf{x}) = 0$ for $|\mathbf{x}| \leqslant a$ and that $\phi_1(\mathbf{x}) = \Phi$, a constant, on $|\mathbf{x}| = a$. Use the results of (a) and (b) to show that

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_{r>a} \frac{\rho_1(\mathbf{x})}{r}\,\mathrm{d}V \ .$$

[*You may assume that* $\phi_1 \to 0$ *as* $|\mathbf{x}| \to \infty$ *sufficiently rapidly that any integrals over the 'sphere at infinity' in* (2) *are zero.*]

Paper 3, Section II
12C Vector Calculus

The vector fields $\mathbf{A}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ obey the evolution equations

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \boldsymbol{\nabla}\psi \ , \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{B} \ , \tag{2}$$

where $\mathbf{u}$ is a given vector field and ψ is a given scalar field. Use suffix notation to show that the scalar field $h = \mathbf{A}\cdot\mathbf{B}$ obeys an evolution equation of the form

$$\frac{\partial h}{\partial t} = \mathbf{B}\cdot\boldsymbol{\nabla} f - \mathbf{u}\cdot\boldsymbol{\nabla} h \ ,$$

where the scalar field f should be identified.

Suppose that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$. Show that, if $\mathbf{u}\cdot\mathbf{n} = \mathbf{B}\cdot\mathbf{n} = 0$ on the surface S of a fixed volume V with outward normal $\mathbf{n}$, then

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \ , \quad \text{where } H = \int_V h\,\mathrm{d}V \ .$$

Suppose that $\mathbf{A} = ar^2 \sin\theta\,\mathbf{e}_\theta + r(a^2 - r^2)\sin\theta\,\mathbf{e}_\phi$ with respect to spherical polar coordinates, and that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Show that

$$h = ar^2(a^2 + r^2)\sin^2\theta \ ,$$

and calculate the value of H when V is the sphere $r \leqslant a$.

$$\left[\ \textit{In spherical polar coordinates } \boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}. \ \right]$$