Paper 3, Section I

3C Vector Calculus

Consider the vector field

$$\mathbf{F} = (-y/(x^2+y^2),\, x/(x^2+y^2),\, 0)$$

defined on all of $\mathbb{R}^3$ except the z axis. Compute $\nabla \times \mathbf{F}$ on the region where it is defined.

Let γ_1 be the closed curve defined by the circle in the xy-plane with centre $(2,2,0)$ and radius 1, and γ_2 be the closed curve defined by the circle in the xy-plane with centre $(0,0,0)$ and radius 1.

By using your earlier result, or otherwise, evaluate the line integral $\oint_{\gamma_1} \mathbf{F} \cdot d\mathbf{x}$.

By explicit computation, evaluate the line integral $\oint_{\gamma_2} \mathbf{F} \cdot d\mathbf{x}$. Is your result consistent with Stokes' theorem? Explain your answer briefly.

Paper 3, Section I

4C Vector Calculus

A curve in two dimensions is defined by the parameterised Cartesian coordinates

$$x(u) = ae^{bu}\cos u\,, \qquad y(u) = ae^{bu}\sin u\,,$$

where the constants $a, b > 0$. Sketch the curve segment corresponding to the range $0 \leqslant u \leqslant 3\pi$. What is the length of the curve segment between the points $(x(0), y(0))$ and $(x(U), y(U))$, as a function of U?

A geometrically sensitive ant walks along the curve with varying speed $\kappa(u)^{-1}$, where $\kappa(u)$ is the curvature at the point corresponding to parameter u. Find the time taken by the ant to walk from $(x(2n\pi), y(2n\pi))$ to $(x(2(n+1)\pi), y(2(n+1)\pi))$, where n is a positive integer, and hence verify that this time is independent of n.

[*You may quote without proof the formula* $\kappa(u) = \dfrac{|x'(u)y''(u) - y'(u)x''(u)|}{((x'(u))^2 + (y'(u))^2)^{3/2}}$.]

Paper 3, Section II

9C Vector Calculus

(a) Define a rank two tensor and show that if two rank two tensors A_{ij} and B_{ij} are the same in one Cartesian coordinate system, then they are the same in all Cartesian coordinate systems.

The quantity C_{ij} has the property that, for every rank two tensor A_{ij}, the quantity $C_{ij}A_{ij}$ is a scalar. Is C_{ij} necessarily a rank two tensor? Justify your answer with a proof from first principles, or give a counterexample.

(b) Show that, if a tensor T_{ij} is invariant under rotations about the x_3-axis, then it has the form

$$\begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} .$$

(c) The *inertia tensor* about the origin of a rigid body occupying volume V and with variable mass density $\rho(\mathbf{x})$ is defined to be

$$I_{ij} = \int_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\, \mathrm{d}V .$$

The rigid body B has uniform density ρ and occupies the cylinder

$$\{ (x_1, x_2, x_3) \, : \, -2 \leqslant x_3 \leqslant 2, \, x_1^2 + x_2^2 \leqslant 1 \} .$$

Show that the inertia tensor of B about the origin is diagonal in the (x_1, x_2, x_3) coordinate system, and calculate its diagonal elements.

Paper 3, Section II
10C Vector Calculus

Let $f(x,y)$ be a function of two variables, and R a region in the xy-plane. State the rule for evaluating $\int_R f(x,y)\,\mathrm{d}x\mathrm{d}y$ as an integral with respect to new variables $u(x,y)$ and $v(x,y)$.

Sketch the region R in the xy-plane defined by

$$R = \{\,(x,y) : x^2+y^2 \leqslant 2\,,\; x^2-y^2 \geqslant 1\,,\; x \geqslant 0\,,\; y \geqslant 0\,\}\,.$$

Sketch the corresponding region in the uv-plane, where

$$u = x^2+y^2\,, \qquad v = x^2-y^2\,.$$

Express the integral

$$I = \int_R (x^5 y - x y^5)\exp(4x^2y^2)\,\mathrm{d}x\,\mathrm{d}y$$

as an integral with respect to u and v. Hence, or otherwise, calculate I.

Paper 3, Section II
11C Vector Calculus

State the divergence theorem (also known as Gauss' theorem) relating the surface and volume integrals of appropriate fields.

The surface S_1 is defined by the equation $z = 3 - 2\,x^2 - 2\,y^2$ for $1 \leqslant z \leqslant 3$; the surface S_2 is defined by the equation $x^2+y^2=1$ for $0 \leqslant z \leqslant 1$; the surface S_3 is defined by the equation $z=0$ for x,y satisfying $x^2+y^2 \leqslant 1$. The surface S is defined to be the union of the surfaces S_1, S_2 and S_3. Sketch the surfaces S_1, S_2, S_3 and (hence) S.

The vector field $\mathbf{F}$ is defined by

$$\mathbf{F}(x,y,z) = (\,xy + x^6,\, -\tfrac{1}{2}y^2 + y^8,\, z\,)\,.$$

Evaluate the integral

$$\oint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}\,,$$

where the surface element $\mathrm{d}\mathbf{S}$ points in the direction of the outward normal to S.

Paper 3, Section II

12C Vector Calculus

Given a spherically symmetric mass distribution with density ρ, explain how to obtain the gravitational field $\mathbf{g} = -\boldsymbol{\nabla}\phi$, where the potential ϕ satisfies Poisson's equation

$$\nabla^2\phi = 4\pi G\rho\,.$$

The remarkable planet Geometria has radius 1 and is composed of an infinite number of stratified spherical shells S_n labelled by integers $n \geqslant 1$. The shell S_n has uniform density $2^{n-1}\rho_0$, where ρ_0 is a constant, and occupies the volume between radius 2^{-n+1} and 2^{-n}.

Obtain a closed form expression for the mass of Geometria.

Obtain a closed form expression for the gravitational field $\mathbf{g}$ due to Geometria at a distance $r = 2^{-N}$ from its centre of mass, for each positive integer $N \geqslant 1$. What is the potential $\phi(r)$ due to Geometria for $r > 1$?