Paper 3, Section I

3B Vector Calculus

What does it mean for a vector field $\mathbf{F}$ to be *irrotational* ?

The field $\mathbf{F}$ is irrotational and $\mathbf{x}_0$ is a given point. Write down a scalar potential $V(\mathbf{x})$ with $\mathbf{F} = -\boldsymbol{\nabla} V$ and $V(\mathbf{x}_0) = 0$. Show that this potential is well defined.

For what value of m is the field $\dfrac{\cos\theta\cos\phi}{r}\,\mathbf{e}_\theta + \dfrac{m\sin\phi}{r}\,\mathbf{e}_\phi$ irrotational, where (r, θ, ϕ) are spherical polar coordinates? What is the corresponding potential $V(\mathbf{x})$ when $\mathbf{x}_0$ is the point $r = 1, \theta = 0$?

$$\left[\textit{In spherical polar coordinates } \boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta\,F_\phi \end{vmatrix} . \right]$$

Paper 3, Section I

4B Vector Calculus

State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$, where $r = |\mathbf{x}|$.

A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{\mathbf{k}}{r} + \frac{(\mathbf{k}\cdot\mathbf{x})\mathbf{x}}{r^3},$$

where $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$ using suffix notation, and show that d_{ij} splits naturally into symmetric and antisymmetric parts. Deduce that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and that

$$\boldsymbol{\nabla}\times\mathbf{u} = \frac{2\mathbf{k}\times\mathbf{x}}{r^3} .$$

Paper 3, Section II
9B Vector Calculus

Let S be a bounded region of $\mathbb{R}^2$ and ∂S be its boundary. Let u be the unique solution to Laplace's equation in S, subject to the boundary condition $u = f$ on ∂S, where f is a specified function. Let w be any smooth function with $w = f$ on ∂S. By writing $w = u + \delta$, or otherwise, show that

$$\int_S |\nabla w|^2 \, \mathrm{d}A \geqslant \int_S |\nabla u|^2 \, \mathrm{d}A \, . \qquad (*)$$

Let S be the unit disc in $\mathbb{R}^2$. By considering functions of the form $g(r)\cos\theta$ on both sides of $(*)$, where r and θ are polar coordinates, deduce that

$$\int_0^1 \left(r\left(\frac{\mathrm{d}g}{\mathrm{d}r}\right)^2 + \frac{g^2}{r} \right) \mathrm{d}r \geqslant 1$$

for any differentiable function $g(r)$ satisfying $g(1) = 1$ and for which the integral converges at $r = 0$.

$$\left[\nabla f(r,\theta) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta} \right), \qquad \nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \, . \right]$$

Paper 3, Section II
10B Vector Calculus

Give a necessary condition for a given vector field $\mathbf{J}$ to be the curl of another vector field $\mathbf{B}$. Is the vector field $\mathbf{B}$ unique? If not, explain why not.

State Stokes' theorem and use it to evaluate the area integral

$$\int_S \left(y^2, z^2, x^2\right) \cdot \mathbf{dA} \, ,$$

where S is the half of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

that lies in $z \geqslant 0$, and the area element $\mathbf{dA}$ points out of the ellipsoid.

Paper 3, Section II

11B Vector Calculus

A second-rank tensor $T(\mathbf{y})$ is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j)|\mathbf{y} - \mathbf{x}|^{2n-2}\, \mathrm{d}A(\mathbf{x})\,,$$

where $\mathbf{y}$ is a fixed vector with $|\mathbf{y}| = a$, $n > -1$, and the integration is over all points $\mathbf{x}$ lying on the surface S of the sphere of radius a, centred on the origin. Explain briefly why T might be expected to have the form

$$T_{ij} \;=\; \alpha\delta_{ij} + \beta y_i\, y_j\,,$$

where α and β are scalar constants.

Show that $\mathbf{y}\cdot(\mathbf{y} - \mathbf{x}) = a^2(1 - \cos\theta)$, where θ is the angle between $\mathbf{y}$ and $\mathbf{x}$, and find a similar expression for $|\mathbf{y} - \mathbf{x}|^2$. Using suitably chosen spherical polar coordinates, show that

$$y_i T_{ij}\, y_j \;=\; \frac{\pi a^2 (2a)^{2n+2}}{n+2}\,.$$

Hence, by evaluating another scalar integral, determine α and β, and find the value of n for which T is isotropic.

Paper 3, Section II

12B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{x})$ in a region V of $\mathbb{R}^3$ bounded by a smooth surface S.

Let $f(x, y, z)$ be a homogeneous function of degree n, that is, $f(kx, ky, kz) = k^n f(x, y, z)$ for any real number k. By differentiating with respect to k, show that

$$\mathbf{x}\cdot\boldsymbol{\nabla} f = nf\,.$$

Deduce that

$$\int_V f\,\mathrm{d}V = \frac{1}{n+3}\int_S f\,\mathbf{x}\cdot\mathbf{dA}\,. \qquad (\dagger)$$

Let V be the cone $0 \leqslant z \leqslant \alpha$, $\alpha\sqrt{x^2+y^2} \leqslant z$, where α is a positive constant. Verify that $(\dagger)$ holds for the case $f = z^4 + \alpha^4(x^2+y^2)^2$.