

3/I/3C Vector Calculus

A curve is given in terms of a parameter t by

$$\mathbf{x}(t) = \left(t - \frac{1}{3}t^3, t^2, t + \frac{1}{3}t^3\right).$$

- (i) Find the arc length of the curve between the points with $t = 0$ and $t = 1$.
- (ii) Find the unit tangent vector at the point with parameter t , and show that the principal normal is orthogonal to the z direction at each point on the curve.

3/I/4C Vector Calculus

What does it mean to say that T_{ij} transforms as a second rank tensor?

If T_{ij} transforms as a second rank tensor, show that $\frac{\partial T_{ij}}{\partial x_j}$ transforms as a vector.

3/II/9C Vector Calculus

Let $\mathbf{F} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$, where \mathbf{x} is the position vector and $\boldsymbol{\omega}$ is a uniform vector field.

- (i) Use the divergence theorem to evaluate the surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, where S is the closed surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.
- (ii) Show that $\nabla \times \mathbf{F} = 0$. Show further that the scalar field ϕ given by

$$\phi = \frac{1}{2} (\boldsymbol{\omega} \cdot \mathbf{x})^2 - \frac{1}{2} (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{x} \cdot \mathbf{x})$$

satisfies $\mathbf{F} = \nabla\phi$. Describe geometrically the surfaces of constant ϕ .

3/II/10C Vector Calculus

Find the effect of a rotation by $\pi/2$ about the z -axis on the tensor

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}.$$

Hence show that the most general isotropic tensor of rank 2 is $\lambda\delta_{ij}$, where λ is an arbitrary scalar.

Prove that there is no non-zero isotropic vector, and write down without proof the most general isotropic tensor of rank 3.

Deduce that if T_{ijkl} is an isotropic tensor then the following results hold, for some scalars μ and ν :

- (i) $\epsilon_{ijk} T_{ijkl} = 0$;
- (ii) $\delta_{ij} T_{ijkl} = \mu \delta_{kl}$;
- (iii) $\epsilon_{ijm} T_{ijkl} = \nu \epsilon_{klm}$.

Verify these three results in the case $T_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$, expressing μ and ν in terms of α , β and γ .

3/II/11C Vector Calculus

Let V be a volume in \mathbb{R}^3 bounded by a closed surface S .

(a) Let f and g be twice differentiable scalar fields such that $f = 1$ on S and $\nabla^2 g = 0$ in V . Show that

$$\int_V \nabla f \cdot \nabla g \, dV = 0.$$

(b) Let V be the sphere $|\mathbf{x}| \leq a$. Evaluate the integral

$$\int_V \nabla u \cdot \nabla v \, dV$$

in the cases where u and v are given in spherical polar coordinates by:

- (i) $u = r$, $v = r \cos \theta$;
- (ii) $u = r/a$, $v = r^2 \cos^2 \theta$;
- (iii) $u = r/a$, $v = 1/r$.

Comment on your results in the light of part (a).

3/II/12C **Vector Calculus**

Let A be the closed planar region given by

$$y \leq x \leq 2y, \quad \frac{1}{y} \leq x \leq \frac{2}{y}.$$

(i) Evaluate by means of a suitable change of variables the integral

$$\int_A \frac{x}{y} dx dy.$$

(ii) Let C be the boundary of A . Evaluate the line integral

$$\oint_C \frac{x^2}{2y} dy - dx$$

by integrating along each section of the boundary.

(iii) Comment on your results.