3/I/3A Vector Calculus

(i) Give definitions for the unit tangent vector $\hat{\mathbf{T}}$ and the curvature κ of a parametrised curve $\mathbf{x}(t)$ in $\mathbb{R}^3$. Calculate $\hat{\mathbf{T}}$ and κ for the circular helix

$$\mathbf{x}(t) = (a\cos t\,,\ a\sin t\,,\ bt)\,,$$

where a and b are constants.

(ii) Find the normal vector and the equation of the tangent plane to the surface S in $\mathbb{R}^3$ given by

$$z = x^2y^3 - y + 1$$

at the point $x = 1$, $y = 1$, $z = 1$.

3/I/4A Vector Calculus

By using suffix notation, prove the following identities for the vector fields $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})\,;$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})\,.$$

3/II/9A Vector Calculus

(i) Define what is meant by a conservative vector field. Given a vector field $\mathbf{A} = (A_1(x,y), A_2(x,y))$ and a function $\psi(x,y)$ defined in $\mathbb{R}^2$, show that, if $\psi\mathbf{A}$ is a conservative vector field, then

$$\psi\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}\right) = A_2\frac{\partial \psi}{\partial x} - A_1\frac{\partial \psi}{\partial y}\,.$$

(ii) Given two functions $P(x,y)$ and $Q(x,y)$ defined in $\mathbb{R}^2$, prove Green's theorem,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,,$$

where C is a simple closed curve bounding a region R in $\mathbb{R}^2$.

Through an appropriate choice for P and Q, find an expression for the area of the region R, and apply this to evaluate the area of the ellipse bounded by the curve

$$x = a\cos\theta\,, \qquad y = b\sin\theta\,, \qquad 0 \le \theta \le 2\pi\,.$$

3/II/10A **Vector Calculus**

For a given charge distribution $\rho(x,y,z)$ and divergence-free current distribution $\mathbf{J}(x,y,z)$ (i.e. $\nabla\cdot\mathbf{J}=0$) in $\mathbb{R}^3$, the electric and magnetic fields $\mathbf{E}(x,y,z)$ and $\mathbf{B}(x,y,z)$ satisfy the equations

$$\nabla\times\mathbf{E}=0\,,\qquad \nabla\cdot\mathbf{B}=0\,,\qquad \nabla\cdot\mathbf{E}=\rho\,,\qquad \nabla\times\mathbf{B}=\mathbf{J}\,.$$

The radiation flux vector $\mathbf{P}$ is defined by $\mathbf{P}=\mathbf{E}\times\mathbf{B}$.

For a closed surface S around a region V, show using Gauss' theorem that the flux of the vector $\mathbf{P}$ through S can be expressed as

$$\iint_S \mathbf{P}\cdot\mathbf{dS} = -\iiint_V \mathbf{E}\cdot\mathbf{J}\,dV\,. \qquad (*)$$

For electric and magnetic fields given by

$$\mathbf{E}(x,y,z)=(z,\ 0,\ x)\,,\qquad \mathbf{B}(x,y,z)=(0,\ -xy,\ xz)\,,$$

find the radiation flux through the quadrant of the unit spherical shell given by

$$x^2+y^2+z^2=1\,,\quad \text{with}\quad 0\le x\le 1\,,\quad 0\le y\le 1\,,\quad -1\le z\le 1\,.$$

[*If you use (∗), note that an open surface has been specified.*]

3/II/11A Vector Calculus

The function $\phi(x, y, z)$ satisfies $\nabla^2\phi = 0$ in V and $\phi = 0$ on S, where V is a region of $\mathbb{R}^3$ which is bounded by the surface S. Prove that $\phi = 0$ everywhere in V.

Deduce that there is at most one function $\psi(x, y, z)$ satisfying $\nabla^2\psi = \rho$ in V and $\psi = f$ on S, where $\rho(x, y, z)$ and $f(x, y, z)$ are given functions.

Given that the function $\psi = \psi(r)$ depends only on the radial coordinate $r = |\mathbf{x}|$, use Cartesian coordinates to show that

$$\nabla\psi = \frac{1}{r}\frac{d\psi}{dr}\mathbf{x}\,, \qquad \nabla^2\psi = \frac{1}{r}\frac{d^2(r\psi)}{dr^2}\,.$$

Find the general solution in this radial case for $\nabla^2\psi = c$ where c is a constant.

Find solutions $\psi(r)$ for a solid sphere of radius $r = 2$ with a central cavity of radius $r = 1$ in the following three regions:

(i) $0 \leqslant r \leqslant 1$ where $\nabla^2\psi = 0$ and $\psi(1) = 1$ and ψ bounded as $r \to 0$;

(ii) $1 \leqslant r \leqslant 2$ where $\nabla^2\psi = 1$ and $\psi(1) = \psi(2) = 1$;

(iii) $r \geqslant 2$ where $\nabla^2\psi = 0$ and $\psi(2) = 1$ and $\psi \to 0$ as $r \to \infty$.

3/II/12A Vector Calculus

Show that any second rank Cartesian tensor P_{ij} in $\mathbb{R}^3$ can be written as a sum of a symmetric tensor and an antisymmetric tensor. Further, show that P_{ij} can be decomposed into the following terms

$$P_{ij} = P\delta_{ij} + S_{ij} + \epsilon_{ijk}A_k\,, \qquad (\dagger)$$

where S_{ij} is symmetric and traceless. Give expressions for P, S_{ij} and A_k explicitly in terms of P_{ij}.

For an isotropic material, the stress P_{ij} can be related to the strain T_{ij} through the stress–strain relation, $P_{ij} = c_{ijkl}T_{kl}$, where the elasticity tensor is given by

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

and α, β and γ are scalars. As in (†), the strain T_{ij} can be decomposed into its trace T, a symmetric traceless tensor W_{ij} and a vector V_k. Use the stress–strain relation to express each of T, W_{ij} and V_k in terms of P, S_{ij} and A_k.

Hence, or otherwise, show that if T_{ij} is symmetric then so is P_{ij}. Show also that the stress-strain relation can be written in the form

$$P_{ij} = \lambda\,\delta_{ij}T_{kk} + \mu\,T_{ij}\,,$$

where μ and λ are scalars.