

**3/I/3A Vector Calculus**

Consider the vector field  $\mathbf{F}(\mathbf{x}) = ((3x^3 - x^2)y, (y^3 - 2y^2 + y)x, z^2 - 1)$  and let  $S$  be the surface of a unit cube with one corner at  $(0, 0, 0)$ , another corner at  $(1, 1, 1)$  and aligned with edges along the  $x$ -,  $y$ - and  $z$ -axes. Use the divergence theorem to evaluate

$$I = \int_S \mathbf{F} \cdot d\mathbf{S}.$$

Verify your result by calculating the integral directly.

**3/I/4A Vector Calculus**

Use suffix notation in Cartesian coordinates to establish the following two identities for the vector field  $\mathbf{v}$ :

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0, \quad (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}).$$

**3/II/9A Vector Calculus**

Evaluate the line integral

$$\int \alpha(x^2 + xy)dx + \beta(x^2 + y^2)dy,$$

with  $\alpha$  and  $\beta$  constants, along each of the following paths between the points  $A = (1, 0)$  and  $B = (0, 1)$ :

- (i) the straight line between  $A$  and  $B$ ;
- (ii) the  $x$ -axis from  $A$  to the origin  $(0, 0)$  followed by the  $y$ -axis to  $B$ ;
- (iii) anti-clockwise from  $A$  to  $B$  around the circular path centred at the origin  $(0, 0)$ .

You should obtain the same answer for the three paths when  $\alpha = 2\beta$ . Show that when  $\alpha = 2\beta$ , the integral takes the same value along *any* path between  $A$  and  $B$ .

### 3/II/10A Vector Calculus

State Stokes' theorem for a vector field  $\mathbf{A}$ .

By applying Stokes' theorem to the vector field  $\mathbf{A} = \phi \mathbf{k}$ , where  $\mathbf{k}$  is an arbitrary constant vector in  $\mathbb{R}^3$  and  $\phi$  is a scalar field defined on a surface  $S$  bounded by a curve  $\partial S$ , show that

$$\int_S d\mathbf{S} \times \nabla \phi = \int_{\partial S} \phi d\mathbf{x}.$$

For the vector field  $\mathbf{A} = x^2 y^4 (1, 1, 1)$  in Cartesian coordinates, evaluate the line integral

$$I = \int \mathbf{A} \cdot d\mathbf{x},$$

around the boundary of the quadrant of the unit circle lying between the  $x$ - and  $y$ -axes, that is, along the straight line from  $(0, 0, 0)$  to  $(1, 0, 0)$ , then the circular arc  $x^2 + y^2 = 1$ ,  $z = 0$  from  $(1, 0, 0)$  to  $(0, 1, 0)$  and finally the straight line from  $(0, 1, 0)$  back to  $(0, 0, 0)$ .

### 3/II/11A Vector Calculus

In a region  $R$  of  $\mathbb{R}^3$  bounded by a closed surface  $S$ , suppose that  $\phi_1$  and  $\phi_2$  are both solutions of  $\nabla^2 \phi = 0$ , satisfying boundary conditions on  $S$  given by  $\phi = f$  on  $S$ , where  $f$  is a given function. Prove that  $\phi_1 = \phi_2$ .

In  $\mathbb{R}^2$  show that

$$\phi(x, y) = (a_1 \cosh \lambda x + a_2 \sinh \lambda x)(b_1 \cos \lambda y + b_2 \sin \lambda y)$$

is a solution of  $\nabla^2 \phi = 0$ , for any constants  $a_1, a_2, b_1, b_2$  and  $\lambda$ . Hence, or otherwise, find a solution  $\phi(x, y)$  in the region  $x \geq 0$  and  $0 \leq y \leq a$  which satisfies:

$$\begin{aligned} \phi(x, 0) = 0, \quad \phi(x, a) = 0, \quad x \geq 0, \\ \phi(0, y) = \sin \frac{n\pi y}{a}, \quad \phi(x, y) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad 0 \leq y \leq a, \end{aligned}$$

where  $a$  is a real constant and  $n$  is an integer.

### 3/II/12A Vector Calculus

Define what is meant by an isotropic tensor. By considering a rotation of a second rank isotropic tensor  $B_{ij}$  by  $90^\circ$  about the  $z$ -axis, show that its components must satisfy  $B_{11} = B_{22}$  and  $B_{13} = B_{31} = B_{23} = B_{32} = 0$ . Now consider a second and different rotation to show that  $B_{ij}$  must be a multiple of the Kronecker delta,  $\delta_{ij}$ .

Suppose that a homogeneous but anisotropic crystal has the conductivity tensor

$$\sigma_{ij} = \alpha\delta_{ij} + \gamma n_i n_j,$$

where  $\alpha, \gamma$  are real constants and the  $n_i$  are the components of a constant unit vector  $\mathbf{n}$  ( $\mathbf{n} \cdot \mathbf{n} = 1$ ). The electric current density  $\mathbf{J}$  is then given in components by

$$J_i = \sigma_{ij} E_j,$$

where  $E_j$  are the components of the electric field  $\mathbf{E}$ . Show that

- (i) if  $\alpha \neq 0$  and  $\gamma \neq 0$ , then there is a plane such that if  $\mathbf{E}$  lies in this plane, then  $\mathbf{E}$  and  $\mathbf{J}$  must be parallel, and
- (ii) if  $\gamma \neq -\alpha$  and  $\alpha \neq 0$ , then  $\mathbf{E} \neq 0$  implies  $\mathbf{J} \neq 0$ .

If  $D_{ij} = \epsilon_{ijk} n_k$ , find the value of  $\gamma$  such that

$$\sigma_{ij} D_{jk} D_{km} = -\sigma_{im}.$$