

3/I/3A Vector Calculus

Let $\mathbf{A}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ be time-dependent, continuously differentiable vector fields on \mathbb{R}^3 satisfying

$$\frac{\partial \mathbf{A}}{\partial t} = \nabla \times \mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{A}.$$

Show that for any bounded region V ,

$$\frac{d}{dt} \left[\frac{1}{2} \int_V (\mathbf{A}^2 + \mathbf{B}^2) dV \right] = - \int_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{S},$$

where S is the boundary of V .

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Given a curve $\gamma(s)$ in \mathbb{R}^3 , parameterised such that $\|\gamma'(s)\| = 1$ and with $\gamma''(s) \neq 0$, define the tangent $\mathbf{t}(s)$, the principal normal $\mathbf{p}(s)$, the curvature $\kappa(s)$ and the binormal $\mathbf{b}(s)$.

The torsion $\tau(s)$ is defined by

$$\tau = -\mathbf{b}' \cdot \mathbf{p}.$$

Sketch a circular helix showing $\mathbf{t}, \mathbf{p}, \mathbf{b}$ and \mathbf{b}' at a chosen point. What is the sign of the torsion for your helix? Sketch a second helix with torsion of the opposite sign.

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Let V be a bounded region of \mathbb{R}^3 and S be its boundary. Let ϕ be the unique solution to $\nabla^2\phi = 0$ in V , with $\phi = f(\mathbf{x})$ on S , where f is a given function. Consider any smooth function w also equal to $f(\mathbf{x})$ on S . Show, by using Green's first theorem or otherwise, that

$$\int_V |\nabla w|^2 dV \geq \int_V |\nabla\phi|^2 dV.$$

[Hint: Set $w = \phi + \delta$.]

Consider the partial differential equation

$$\frac{\partial}{\partial t} w = \nabla^2 w,$$

for $w(t, \mathbf{x})$, with initial condition $w(0, \mathbf{x}) = w_0(\mathbf{x})$ in V , and boundary condition $w(t, \mathbf{x}) = f(\mathbf{x})$ on S for all $t \geq 0$. Show that

$$\frac{\partial}{\partial t} \int_V |\nabla w|^2 dV \leq 0, \quad (*)$$

with equality holding only when $w(t, \mathbf{x}) = \phi(\mathbf{x})$.

Show that (*) remains true with the boundary condition

$$\frac{\partial w}{\partial t} + \alpha(\mathbf{x}) \frac{\partial w}{\partial n} = 0$$

on S , provided $\alpha(\mathbf{x}) \geq 0$.

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Write down Stokes' theorem for a vector field $\mathbf{B}(\mathbf{x})$ on \mathbb{R}^3 .

Consider the bounded surface S defined by

$$z = x^2 + y^2, \quad \frac{1}{4} \leq z \leq 1.$$

Sketch the surface and calculate the surface element $d\mathbf{S}$. For the vector field

$$\mathbf{B} = (-y^3, x^3, z^3),$$

calculate $I = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$ directly.

Show using Stokes' theorem that I may be rewritten as a line integral and verify this yields the same result.

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Explain, with justification, the significance of the eigenvalues of the Hessian in classifying the critical points of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. In what circumstances are the eigenvalues inconclusive in establishing the character of a critical point?

Consider the function on \mathbb{R}^2 ,

$$f(x, y) = xye^{-\alpha(x^2+y^2)}.$$

Find and classify all of its critical points, for all real α . How do the locations of the critical points change as $\alpha \rightarrow 0$?

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Express the integral

$$I = \int_0^\infty dx \int_0^1 dy \int_0^x dz xe^{-Ax/y - Bxy - Cyz}$$

in terms of the new variables $\alpha = x/y$, $\beta = xy$, and $\gamma = yz$. Hence show that

$$I = \frac{1}{2A(A+B)(A+B+C)}.$$

You may assume A, B and C are positive. [*Hint: Remember to calculate the limits of the integral.*]