### 3/I/3A **Vector Calculus**

Sketch the curve $y^2 = x^2 + 1$. By finding a parametric representation, or otherwise, determine the points on the curve where the radius of curvature is least, and compute its value there.

[*Hint: you may use the fact that the radius of curvature of a parametrized curve* $(x(t), y(t))$ *is* $(\dot{x}^2 + \dot{y}^2)^{3/2}/|\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.]

### 3/I/4A **Vector Calculus**

Suppose $V$ is a region in $\mathbb{R}^3$, bounded by a piecewise smooth closed surface $S$, and $\phi(\mathbf{x})$ is a scalar field satisfying

$$\begin{aligned} \boldsymbol{\nabla}^2 \phi &= 0 \quad \text{in } V, \\ \text{and} \quad \phi &= f(\mathbf{x}) \quad \text{on } S\,. \end{aligned}$$

Prove that $\phi$ is determined uniquely in $V$.

How does the situation change if the normal derivative of $\phi$ rather than $\phi$ itself is specified on $S$?

### 3/II/9A **Vector Calculus**

Let $C$ be the closed curve that is the boundary of the triangle $T$ with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Specify a direction along $C$ and consider the integral

$$\oint_C \mathbf{A} \cdot d\mathbf{x}\ ,$$

where $\mathbf{A} = (z^2 - y^2,\ x^2 - z^2,\ y^2 - x^2)$. Explain why the contribution to the integral is the same from each edge of $C$, and evaluate the integral.

State Stokes's theorem and use it to evaluate the surface integral

$$\int_T (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S}\ ,$$

the components of the normal to $T$ being positive.

Show that $d\mathbf{S}$ in the above surface integral can be written in the form $(1,1,1)\,dy\,dz$. Use this to verify your result by a direct calculation of the surface integral.

3/II/10A **Vector Calculus**

Write down an expression for the Jacobian $J$ of a transformation

$$(x, y, z) \to (u, v, w)\,.$$

Use it to show that

$$\int_D f \; dx\, dy\, dz = \int_\Delta \phi \; |J| \; du\, dv\, dw$$

where $D$ is mapped one-to-one onto $\Delta$, and

$$\phi(u, v, w) \;=\; f\big(x(u, v, w),\; y(u, v, w),\; z(u, v, w)\big)\,.$$

Find a transformation that maps the ellipsoid $D$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\,,$$

onto a sphere. Hence evaluate

$$\int_D x^2 \, dx\, dy\, dz\,.$$

3/II/11A **Vector Calculus**

(a) Prove the identity

$$\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) \;=\; (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G} \;+\; (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} \;+\; \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G}) \;+\; \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})\,.$$

(b) If $\mathbf{E}$ is an irrotational vector field ($\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\boldsymbol{\nabla}\phi$.

Show that

$$(2xy^2ze^{-x^2z},\; -2ye^{-x^2z},\; x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential $\phi$.

3/II/12A **Vector Calculus**

State the divergence theorem. By applying this to $f(\mathbf{x})\mathbf{k}$, where $f(\mathbf{x})$ is a scalar field in a closed region $V$ in $\mathbb{R}^3$ bounded by a piecewise smooth surface $S$, and $\mathbf{k}$ an arbitrary constant vector, show that

$$\int_V \boldsymbol{\nabla} f\, dV = \int_S f\, d\mathbf{S}\,. \qquad (*)$$

A vector field $\mathbf{G}$ satisfies

$$\boldsymbol{\nabla}\cdot\mathbf{G} = \rho(\mathbf{x})$$
$$\text{with}\quad \rho(\mathbf{x}) = \begin{cases} \rho_0 & |\mathbf{x}| \leqslant a \\ 0 & |\mathbf{x}| > a\,. \end{cases}$$

By applying the divergence theorem to $\displaystyle\int_V \boldsymbol{\nabla}\cdot\mathbf{G}\, dV$, prove Gauss's law

$$\int_S \mathbf{G}\cdot d\mathbf{S} = \int_V \rho(\mathbf{x})\, dV,$$

where $S$ is the piecewise smooth surface bounding the volume $V$.

Consider the spherically symmetric solution

$$\mathbf{G}(\mathbf{x}) = G(r)\,\frac{\mathbf{x}}{r}\ ,$$

where $r = |\mathbf{x}|$. By using Gauss's law with $S$ a sphere of radius $r$, centre $\mathbf{0}$, in the two cases $0 < r \leqslant a$ and $r > a$, show that

$$\mathbf{G}(\mathbf{x}) = \begin{cases} \dfrac{\rho_0}{3}\mathbf{x} & r \leqslant a \\ \dfrac{\rho_0}{3}\left(\dfrac{a}{r}\right)^3 \mathbf{x} & r > a\,. \end{cases}$$

The scalar field $f(\mathbf{x})$ satisfies $\mathbf{G} = \boldsymbol{\nabla} f$. Assuming that $f \to 0$ as $r \to \infty$, and that $f$ is continuous at $r = a$, find $f$ everywhere.

By using a symmetry argument, explain why $(*)$ is clearly satisfied for this $f$ if $S$ is any sphere centred at the origin.