

Example Sheet 4

- 1 If $u_i(\mathbf{x})$ is a vector field, show that $\partial u_i / \partial x_j$ transforms as a second-rank tensor.
If $\sigma_{ij}(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij} / \partial x_j$ transforms as a vector.

- 2 The current J_i due to an electric field E_i is given by $J_i = \sigma_{ij} E_j$, where σ_{ij} is the conductivity tensor. In a certain coordinate system,

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current flow is largest, for an electric field of fixed magnitude.

- 3 Using the transformation law for a second-rank tensor T_{ij} , show that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij} T_{ji}, \quad \gamma = T_{ij} T_{jk} T_{ki}$$

are the same in all Cartesian coordinate systems. If T_{ij} is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are roots of the cubic equation

$$\lambda^3 - \alpha \lambda^2 + \frac{1}{2}(\alpha^2 - \beta) \lambda - \frac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

- 4 Given vectors $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by

$$T_{ij} = u_i v_j + v_i w_j; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k.$$

- 5 The fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ obey Maxwell's equations with zero charge and current. Show that the Poynting vector $\mathbf{P} = \mu_0^{-1} \mathbf{E} \times \mathbf{B}$ satisfies

$$\frac{1}{c^2} \frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{2} \epsilon_0 \delta_{ij} (E_k E_k + c^2 B_k B_k) - \epsilon_0 (E_i E_j + c^2 B_i B_j).$$

- 6 The velocity field $\mathbf{u}(\mathbf{x}, t)$ of an inviscid compressible gas obeys

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{and} \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p$$

where $\rho(\mathbf{x}, t)$ is the density and $p(\mathbf{x}, t)$ is the pressure. Show that

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \rho u^2 u_i + p u_i \right) = p \nabla \cdot \mathbf{u} \quad \text{and} \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (t_{ij}) = 0$$

for a suitable symmetric tensor t_{ij} , to be determined.

7 The components of a second-rank tensor are given by a matrix A . Show that

$$A\mathbf{x} = \alpha\mathbf{x} + \boldsymbol{\omega} \times \mathbf{x} + B\mathbf{x} \quad \text{for all } \mathbf{x},$$

for some scalar α , vector $\boldsymbol{\omega}$, and symmetric traceless matrix B . Find α , $\boldsymbol{\omega}$ and B when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

8 A tensor T_{ijk} is symmetric in its first pair of indices and antisymmetric in its second pair of indices. Show that $T_{ijk} = 0$.

9 A tensor has components T_{ij} with respect to Cartesian coordinates x_i . If the tensor is invariant under arbitrary rotations around the x_3 -axis, show that it must have the form

$$(T_{ij}) = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

10 Evaluate the following integrals over all space, where $\gamma > 0$ and $r^2 = x_p x_p$:

$$(i) \int r^{-3} e^{-\gamma r^2} x_i x_j dV; \quad (ii) \int r^{-5} e^{-\gamma r^2} x_i x_j x_k dV.$$

11 A body is symmetrical under rotations through π about three perpendicular axes. Show that any second-rank tensor calculated as an integral over the body is diagonal in a Cartesian coordinate system defined using the axes of symmetry.

Find the inertia tensor of a cuboid of uniform density about its centre, if its sides have lengths $2a$, $2b$ and $2c$.

12 In linear elasticity, the symmetric second-rank stress tensor σ_{ij} depends on the symmetric second-rank strain tensor e_{kl} according to $\sigma_{ij} = c_{ijkl} e_{kl}$. Explain why c_{ijkl} must be a fourth-rank tensor, assuming $c_{ijkl} = c_{ijlk}$. For an isotropic medium, use the most general possible form for c_{ijkl} (which you may quote) to show that

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where λ and μ are scalars.

Invert this equation to express e_{ij} in terms of σ_{ij} , assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of σ_{ij} and e_{ij} coincide.

The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2} e_{ij} \sigma_{ij}$. Show that E is strictly positive for any non-zero strain e_{ij} provided $\mu > 0$ and $\lambda > -2\mu/3$.

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