Mathematical Tripos Part IA **Lent Term 2010**
Vector Calculus **Dr. J.M. Evans**

Example Sheet 3

1 (i) Write down the operator $\boldsymbol{\nabla}$ in Cartesian coordinates and in spherical polars, and calculate the gradient of

$$\psi \;=\; Ez \;=\; Er\cos\theta$$

in both coordinate systems. By considering the relationship between the basis vectors, check that your answers agree. (ii) Calculate, in three ways, the curl of the vector field

$$\mathbf{A} \;=\; \tfrac{1}{2}B(\,-y\,\mathbf{e}_x + x\,\mathbf{e}_y\,) \;=\; \tfrac{1}{2}B\rho\,\mathbf{e}_\phi \;=\; \tfrac{1}{2}Br\sin\theta\,\mathbf{e}_\phi$$

by applying the standard formulas in Cartesian, cylindrical, and spherical polar coordinates.

2 In cylindrical polar coordinates,

$$\boldsymbol{\nabla} = \mathbf{e}_\rho\,\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\,\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\,\frac{\partial}{\partial z} \qquad \text{and} \qquad \frac{\partial\mathbf{e}_\rho}{\partial\phi} = \mathbf{e}_\phi\,, \quad \frac{\partial\mathbf{e}_\phi}{\partial\phi} = -\mathbf{e}_\rho\,,$$

while all other derivatives of the basis vectors are zero. Derive expressions for $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = A_\rho\mathbf{e}_\rho + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_z$, and also for $\nabla^2\psi$, where ψ is a scalar field.

3 The vector field $\mathbf{B}(\mathbf{x})$ is defined in cylindrical polar coordinates ρ, ϕ, z by

$$\mathbf{B}(\mathbf{x}) \;=\; \rho^{-1}\mathbf{e}_\phi\;, \qquad\qquad \rho \neq 0\;.$$

Calculate $\boldsymbol{\nabla}\times\mathbf{B}$ using the formula for curl in cylindrical polars. Evaluate $\oint_C \mathbf{B}\cdot d\mathbf{x}$, where C is the circle $z=0$, $\rho=1$ and $0 \leqslant \phi \leqslant 2\pi$. Is your answer consistent with Stokes's Theorem?

4 Show that Maxwell's equations for $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ imply that the charge density $\rho(\mathbf{x},t)$ and current density $\mathbf{j}(\mathbf{x},t)$ satisfy the conservation equation $\boldsymbol{\nabla}\cdot\mathbf{j} = -\partial\rho/\partial t$. Show also that if ρ and $\mathbf{j}$ are zero then

$$U \;=\; \tfrac{1}{2}\epsilon_0(\,\mathbf{E}^2 + c^2\mathbf{B}^2\,) \quad \text{and} \quad \mathbf{P} \;=\; \mu_0^{-1}\mathbf{E}\times\mathbf{B} \qquad \text{satisfy} \qquad \boldsymbol{\nabla}\cdot\mathbf{P} \;=\; -\partial U/\partial t\;.$$

5 The scalar field $\varphi(r)$ depends only on $r = |\mathbf{x}|$, where $\mathbf{x}$ is the position vector in three dimensions. Use Cartesian coordinates and the chain rule to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\,\frac{\mathbf{x}}{r}\,, \qquad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\,\varphi'(r)\,.$$

Find the solution of $\nabla^2\varphi = 1$ which is defined on the region $r \leqslant a$ and which satisfies $\varphi(a) = 1$.

6 In two dimensions, write Laplace's equation, $\nabla^2\varphi = 0$, in terms of polar coordinates (r,θ). Verify that there are solutions $\varphi(r,\theta) = A\,r^\alpha\cos n\theta$ for each positive integer n, if α is chosen appropriately. Hence solve Laplace's equation on the following regions, with the given boundary conditions (where V is a constant):

(i) $r \leqslant a\,,\quad \varphi(a,\theta) = V\cos\theta\,;$ (ii) $a \leqslant r \leqslant b\,,\quad \varphi(a,\theta) = 0\,,\quad \varphi(b,\theta) = V\cos 2\theta\;.$

7 Use Gauss's flux method to find the gravitational field $\mathbf{g}(\mathbf{r})$ due to a spherical shell of matter with density

$$\rho(r) = \begin{cases} 0 & \text{for } 0 < r < a, \\ \rho_0 r/a & \text{for } a < r < b, \\ 0 & \text{for } b < r < \infty. \end{cases}$$

Check your answer by calculating the gravitational potential φ directly, by solving Poisson's equation. You should assume that φ is a function only of r, is not singular at the origin, and that $\varphi(r)$ and $\varphi'(r)$ are continuous at $r = a$ and $r = b$.

8 Let $\rho(\mathbf{x})$ be a function on a volume V and $f(\mathbf{x})$ a function on its boundary $S = \partial V$. Show that a solution $\varphi(\mathbf{x})$ to the following problem is unique:

$$\nabla^2 \varphi - \varphi = \rho \ \text{ in } \ V\,, \qquad \frac{\partial \varphi}{\partial n} = f \ \text{ on } \ S\,.$$

9 Show that the solution to Laplace's equation in V with boundary condition

$$g \frac{\partial \varphi}{\partial n} + \varphi = f$$

is unique if $g(\mathbf{x}) \geqslant 0$ on the boundary ∂V.

Find a non-zero solution of Laplace's equation on $|\mathbf{x}| \leqslant 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $|\mathbf{x}| = 1$.

10 The functions $u(\mathbf{x})$ and $v(\mathbf{x})$ on V satisfy $\nabla^2 u = 0$ on V and $v = 0$ on ∂V. Show that

$$\int_V \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \; \mathrm{d}V \;=\; 0\;.$$

Now if $w(\mathbf{x})$ is a function on V with $u = w$ on ∂V, show, by considering $v = w - u$, that

$$\int_V |\boldsymbol{\nabla} w|^2 \; \mathrm{d}V \;\geqslant\; \int_V |\boldsymbol{\nabla} u|^2 \; \mathrm{d}V\,.$$

11 **(a)** An object occupies a volume V with boundary a closed surface S. Let $\psi(\mathbf{x})$ satisfy Laplace's equation in the region outside V, with boundary conditions $\psi = 1$ on S and $\psi \to 0$ as $|\mathbf{x}| \to \infty$. The capacity of the object is then defined to be $C = -\int_S \boldsymbol{\nabla}\psi \cdot \mathrm{d}\mathbf{S}$. Show that the capacity of a sphere of radius R is $4\pi R$.

(b) Suppose the object carries a static distribution of electric charge, with Q the total charge in V, and zero charge outside V. Show that if the electrostatic potential $\varphi(\mathbf{x})$ is a constant φ_0 on S, then $\epsilon_0 C = Q/\varphi_0$.

(c)* Show that the capacity for a cube with edges of length a satisfies $2\pi a < C < 2\sqrt{3}\pi a$. [Hint: Consider a sphere inscribed in the cube, and the cube inscribed in a larger sphere, and use the result of question **10**.]

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