Mathematical Tripos Part IA **Lent Term 2010**
Vector Calculus **Dr. J.M. Evans**

Example Sheet 1

1 Sketch the curve in the plane given parametrically by

$$\mathbf{r}(u) = (\, x(u),\, y(u)\,) = (\, a\cos^3 u,\, a\sin^3 u\,) \qquad \text{with} \qquad 0 \leqslant u \leqslant 2\pi \ .$$

Calculate its tangent vector $\mathrm{d}\mathbf{r}/\mathrm{d}u$ at each point and hence find its total length.

2 In three dimensions, use suffix notation and the summation convention to show that

$$\text{(i)} \quad \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a} \ ; \qquad\qquad \text{(ii)} \quad \boldsymbol{\nabla} r^n = n r^{n-2}\mathbf{x} \ ,$$

where $\mathbf{a}$ is any constant vector and $r = |\mathbf{x}|$.

Given a function $f(\mathbf{r})$ in two dimensions, use the Chain Rule to express its partial derivatives with respect to Cartesian coordinates (x, y) in terms of its partial derivatives with respect to polar coordinates (r, θ). From the relationship between the basis vectors in these coordinate systems, deduce that

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \ .$$

3 Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)\,,$$

along (i) the straight line path joining the origin to $x = y = z = 1$, and (ii) the parabolic path given parametrically by $x = t$, $y = t$, $z = t^2$ with $0 \leqslant t \leqslant 1$.

For which of these integrals do the two paths give the same results, and why?

4 Consider forces $\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the work done, given by the line integrals $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ and $\int \mathbf{G}\cdot\mathrm{d}\mathbf{r}$, along the following paths, each of which consist of straight line segments joining the specified points: (i) $(0,0,0) \to (1,1,1)$; (ii) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$; (iii) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.

5 A curve C is given parametrically in Cartesian coordinates by

$$\mathbf{r}(t) = \big(\cos(\sin nt)\cos t\,,\ \cos(\sin nt)\sin t\,,\ \sin(\sin nt)\big)\,, \qquad 0 \leqslant t \leqslant 2\pi\,,$$

where n is some fixed integer. Using spherical polar coordinates, or otherwise, sketch or describe the curve. Show that

$$\int_C \mathbf{H}\cdot\mathrm{d}\mathbf{r} = 2\pi\ , \qquad \text{where} \qquad \mathbf{H}(\mathbf{r}) = \left(-\frac{y}{x^2+y^2},\ \ \frac{x}{x^2+y^2},\ \ 0\right)$$

and C is traversed in the direction of increasing t. Can $\mathbf{H}(\mathbf{r})$ be written as the gradient of a scalar function? Comment on your results.

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6 Obtain the equation of the plane which is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

7 Use the substitution $x = r\cos\theta$, $y = \frac{1}{2} r \sin\theta$, to evaluate

$$\int_A \frac{x^2}{x^2 + 4y^2}\,\mathrm{d}A\,,$$

where A is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

8 The closed curve C in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area enclosed by C is A. Show, by calculating the integrals explicitly, that

$$\int_C (x^2 y\,\mathrm{d}x + xy^2\,\mathrm{d}y) = \int_A (y^2 - x^2)\,\mathrm{d}A = \frac{104}{105} a^4\,.$$

where C is traversed anticlockwise.

9 The region A is bounded by the segments $x = 0$, $0 \leqslant y \leqslant 1$; $y = 0$, $0 \leqslant x \leqslant 1$; $y = 1$, $0 \leqslant x \leqslant \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1 - x)$. Consider a mapping into the (x, y) plane from the (u, v) plane defined by the transformation $x = u^2 - v^2, \quad y = 2uv$. Sketch A and also the two regions in the (u, v) plane which are mapped into it. Hence evaluate

$$\int_A \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}\,.$$

10 By using a suitable change of variables, calculate the volume within an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\,.$$

11 A tetrahedron V has vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Find the centre of volume, defined by

$$\frac{1}{V}\int_V \mathbf{x}\,\mathrm{d}V\,.$$

12 A solid cone is bounded by the surface $\theta = \alpha$ in spherical polar coordinates and the surface $z = a$. Its mass density is $\rho_0 \cos\theta$. By evaluating a volume integral find the mass of the cone.

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