1 If $\mathbf{u}(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second-rank tensor.
If $\sigma(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij}/\partial x_j$ transforms as a vector.

2 Decompose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

into the form

$$A\mathbf{x} = \alpha\mathbf{x} + \omega \times \mathbf{x} + B\mathbf{x},$$

where α is a scalar, ω a vector and B a traceless symmetric second-rank tensor.

3 The electrical conductivity tensor σ_{ij} has components

$$\sigma_{ij} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Find the direction(s) (i) along which no current flows and (ii) the current is largest.

4 For any second-rank tensor T_{ij}, prove using the transformation law for components that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji} \quad \text{and} \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are the same in all bases.

If T_{ij} is a symmetric tensor, express these invariants in terms of the eigenvalues. Deduce that the cubic equation for the eigenvalues is

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

5 Evaluate the following integrals over the whole of R^3 for positive γ and $r^2 = x_p x_p$:

$$(i) \quad \int r^{-3} e^{-\gamma r^2} x_i x_j \, \mathrm{d}V, \qquad (ii) \quad \int r^{-5} e^{-\gamma r^2} x_i x_j x_k \, \mathrm{d}V.$$

6 A body has the symmetry that its shape is unchanged by arbitrary rotations around a single axis, $\mathbf{e}_3$. Show that any second-rank tensor calculated for the body will take the form

$$\begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

7 A body has the symmetry that its shape in unchanged by rotations of 180^o about three perpendicular axes which form a basis B. Show that any second-rank tensor calculated for the body will be diagonal in B, although the diagonal elements need not be equal.

Evaluate the moments of inertia tensor of a cuboid with sides of length $2a$, $2b$ and $2c$ about the centre of the cuboid.

8 In the piezoelectric effect, an electric polarisation vector P_i is induced by a mechanical stress described by a second order symmetric tensor σ_{jk}. There is a linear relation between them of the form

$$P_i = d_{ijk}\sigma_{jk},$$

where d_{ijk} is a third-rank tensor which depends only on the material. Show that P_i can be non-zero only in an anisotropic material.

9 In linear elasticity, the symmetric second-rank stress tensor σ_{ij} is linear in the symmetric second-rank strain tensor e_{kl}. Show that in an isotropic material

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

with two material constants λ and μ. [*You may quote the form of the general isotropic fourth-rank tensor.*] Solve the above equation to find an expression for e_{ij} in terms of σ_{kl}. Show that the eigenvectors of σ are parallel to the eigenvectors of e.

10* Plastic flow of a granular medium occurs when the magnitude of the tangential force $\mathbf{F}$ exceeds the normal force $\mathbf{N}$ multiplied by the Coulombic friction coefficient μ. At a (symmetric second-rank tensor) stress σ, the internal tangential and normal forces acting across a surface with unit normal $\mathbf{n}$ are calculated as

$$|\mathbf{N}| = n_i\sigma_{ij}n_j, \quad \text{and} \quad |\mathbf{F}| = t_i\sigma_{ij}n_j,$$

where $\mathbf{t}$ is orthogonal to $\mathbf{n}$. Consider only the two-dimensional case and take σ in its principal basis, i.e.

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$

Use $\mathbf{n} = (\cos\theta, \sin\theta)$ and $\mathbf{t} = (-\sin\theta, \cos\theta)$. Find the maximum value of $|\mathbf{F}|/|\mathbf{N}|$ as θ varies. Deduce there will be plastic flow if

$$|\sigma_1 - \sigma_2| > \frac{\mu}{\sqrt{1+\mu^2}}|\sigma_1 + \sigma_2|.$$

I would appreciate any comments and corrections from students and supervisors. Please e-mail `ejh1@cam.ac.uk`.