Mathematical Tripos Part IA — Lent Term 2009

Vector Calculus, Examples sheet 3 — Prof E.J. Hinch

1 If $\boldsymbol{\nabla}.\mathbf{J} = 0$ in the volume V and $\mathbf{J}.\mathbf{n} = 0$ on the surface S which encloses V, show that

$$\int_V \mathbf{J}\,\mathrm{d}V = 0.$$

Hint: use $\frac{\partial}{\partial x_j}(x_i J_j)$.

2 For an electromagnetic field $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, define

$$M_i = \frac{\partial}{\partial x_j}\left(\epsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0 E_k E_k + \frac{1}{\mu_0}B_k B_k\right)\right).$$

Using Maxwell's equations, show that

$$\frac{\partial}{\partial t}\left(\epsilon_0 \mathbf{E}\times\mathbf{B}\right) = \mathbf{M} - \rho\mathbf{E} - \mathbf{J}\times\mathbf{B}.$$

3 Calculate the net flux $\int_S \mathbf{u}.\mathbf{n}\,\mathrm{d}A$ over the hemisphere $z \ge 0$ and $x^2+y^2+z^2 = R^2$ for the flow

$$\mathbf{u} = \left(x(R-z), y(R-z), (R-z)^2\right).$$

Also calculate the net flux through the disk $z = 0$ and $x^2+y^2 \le R^2$. Apply the divergence theorem to show that these two must have the same value.

4 Starting from the equations of conservation of mass and momentum for an inviscid compressible gas,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho\mathbf{u}) = 0 \quad \text{and} \quad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}\right) = -\boldsymbol{\nabla}p + \mathbf{F},$$

derive for a fixed volume V enclosed by a surface S

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2\,\mathrm{d}V + \int_S \tfrac{1}{2}\rho u^2 \mathbf{u}.\mathbf{n}\,\mathrm{d}A = -\int_S p\mathbf{u}.\mathbf{n}\,\mathrm{d}A + \int_V \left(p\boldsymbol{\nabla}.\mathbf{u} + \mathbf{F}.\mathbf{u}\right)\,\mathrm{d}V.$$

5 The scalar function of position ϕ depends only on the radial distance $r = |\mathbf{x}|$, i.e. $\phi = \phi(r)$. Using Cartesian coordinates, show that

$$\boldsymbol{\nabla}\phi = \phi'(r)\frac{\mathbf{x}}{r} \quad \text{and} \quad \nabla^2\phi = \phi''(r) + \frac{2}{r}\phi'(r).$$

Find the solution of $\nabla^2\phi = 1$ in $r \le 1$ which is not singular at the origin and satisfies $\phi = 1$ on $r = 1$.

6 Find, by direct solution of Poisson's equation and by use of Gauss's flux theorem, the gravitational field everywhere due to a spherical shell with density given by

$$\rho(r) = \begin{cases} 0 & \text{for } 0 < r < a, \\ \rho_0 r/a & \text{for } a < r < b, \\ 0 & \text{for } b < r < \infty. \end{cases}$$

You should assume that the potential is a function only of r, is not singular at the origin and that the potential and its first derivative are continuous at $r = a$ and $r = b$.

7 Show that $\phi(\mathbf{x}) = Ar^n \cos\theta$ satisfies Laplace's equation in plane polar coordinates with suitably chosen values of n. Hence solve the problem for $\phi(\mathbf{x})$

$$\begin{aligned} \nabla^2\phi &= 0 \quad \text{in } r \geq a \\ \phi &\to 2r\cos\theta \quad \text{as } r \to \infty \\ \frac{\partial\phi}{\partial r} &= 0 \quad \text{on } r = a. \end{aligned}$$

8 With $z = x + iy$, show that $\phi(x,y) = \text{Re}[f(z)]$ satisfies Laplace's equation. Hence show that $\phi = (x\cos y - y\sin y)e^x$ gives a flow field (Sheet II, question 6)

$$\mathbf{u} = \boldsymbol{\nabla}\phi$$

which is solenoidal. Find the $\psi(x,y)$ which gives this $\mathbf{u} = (\psi_y, -\psi_x, 0)$. What is $\phi + i\psi$?

9 Given $\rho(\mathbf{x})$ in the volume V and $f(\mathbf{x})$ on the surface S which encloses V, show that the solution for $\phi(\mathbf{x})$ is unique to the problem

$$\nabla^2\phi - \phi = \rho \quad \text{in } V \qquad \text{and} \quad \frac{\partial\phi}{\partial n} = f \quad \text{on } S.$$

10 Show that the solution to Laplace's equation with boundary condition

$$\alpha\frac{\partial\phi}{\partial n} + \phi = f$$

is unique (and zero) if $\alpha(\mathbf{x}) \geq 0$. Show, however, that if $f = 0$ and $\alpha = -R$ there is a non-zero (and so non-unique) solution $\phi = Ax$ in the disk $x^2 + y^2 \leq R^2$.

11 Let $u(\mathbf{x})$ be the unique solution of Laplace's equation in the volume V subject to the boundary condition that u is equal to a given function $f(\mathbf{x})$ on the surface S which encloses V. Let v be any function with continuous first partial derivatives in V which vanishes on S. Show that

$$\int_V \boldsymbol{\nabla}u.\boldsymbol{\nabla}v \,\mathrm{d}V = 0.$$

Let w be a function with continuous first partial derivatives in V which satisfies $w = f$ on S. Use the above result with $v = w - u$ to deduce that

$$\int_V |\boldsymbol{\nabla}w|^2 \,\mathrm{d}V \geq \int_V |\boldsymbol{\nabla}u|^2 \,\mathrm{d}V\,,$$

i.e. the solution of the Laplace problem minimises $\int_V |\boldsymbol{\nabla}w|^2 \,\mathrm{d}V$.

12** The capacity C of an object is defined to be the integral over its surface $-\int_S \frac{\partial\phi}{\partial n}\,\mathrm{d}A$, where the potential $\phi(\mathbf{x})$ satisfies Laplace's equation in the volume outside the object, $\phi = 1$ on S and $\phi \to 0$ at ∞. Show that the capacity of a sphere of radius R is $4\pi R$.

Use the previous question to show that a cube with edges of length a has a capacity C bounded by $2\pi a < C < 2\sqrt{3}\pi a$. *[Hint: First relate the minimising integral to the capacity. Then for the lower bound, use the volume outside the inscribing sphere and take w equal to the solution to Laplace's equation outside the cube which is extended by $w = 1$ in the gap between the sphere and the cube.]*

13 Let the surface S enclose the volume V, and let $\mathbf{P}(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$ be two solenoidal vectors ($\boldsymbol{\nabla}.\mathbf{P} = \boldsymbol{\nabla}.\mathbf{Q} = 0$). Show that

$$\int_V \left(\mathbf{Q}.\nabla^2\mathbf{P} - \mathbf{P}.\nabla^2\mathbf{Q}\right)\,\mathrm{d}V = \int_S \left(\mathbf{Q}\times(\boldsymbol{\nabla}\times\mathbf{P}) - \mathbf{P}\times(\boldsymbol{\nabla}\times\mathbf{Q})\right).\mathrm{d}\mathbf{A}\,.$$

I would appreciate any comments and corrections from students and supervisors. Please e-mail `ejh1@cam.ac.uk`.