1 The curve given parametrically by $(a\cos^3 t, a\sin^3 t)$ with $0 \le t \le 2\pi$ is called an *Astroid*. Find its length.

2 The curve defined by $y^2 = x^3$ is called *Neile's parabola*, named after William Neil (1637-1670). Find the length of the segment of Neile's parabola with $0 \le x \le 4$.

3 Find the minimum and maximum curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

4 A circular helix is given by

$$\mathbf{x} = (a\cos t, a\sin t, ct)\,.$$

Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$, binormal $\mathbf{b}$, and torsion τ.

5 Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)\,,$$

along (i) the straight line path joining the origin to $x = y = z = 1$, and (ii) the parabolic path given parametrically by $x = t$, $y = t$, $z = t^2$ with $0 \leqslant t \leqslant 1$.

For which of these integrals do the two paths give the same results, and why?

6 The vector force fields $\mathbf{F}$ and $\mathbf{G}$ are defined by $\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2)$ respectively. (i) Compute the line integrals $\int \mathbf{F}.\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}.\mathrm{d}\mathbf{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. (ii) Compute the line integrals $\int \mathbf{F}.\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}.\mathrm{d}\mathbf{x}$ along the path $\mathbf{x}(t) = (t, t^n, t^2)$ from $(0,0,0)$ to $(1,1,1)$.

7 The closed curve C in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area enclosed by C is A. Show, by calculating the integrals explicitly, that

$$\int_C (x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y) = \int_A (y^2 - x^2)\,\mathrm{d}A = \frac{104}{105}a^4.$$

where C is described anticlockwise.

8 Use the substitution $x = r\cos\theta$, $y = \frac{1}{2}r\sin\theta$, to evaluate

$$\int_A \frac{x^2}{x^2 + 4y^2}\,\mathrm{d}A\,,$$

where A is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

9 The region A is bounded by the segments $x = 0$, $0 \le y \le 1$; $y = 0$, $0 \le x \le 1$; $y = 1$, $0 \le x \le \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1 - x)$. Consider a mapping into the (x, y) plane from the (u, v) plane defined by the transformation $x = u^2 - v^2, \quad y = 2uv$. Sketch A and also the two regions in the (u, v) plane which are mapped into it. Hence evaluate

$$\int_A \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}\,.$$

10 Show without changing the order of integration that

$$\int_0^1 \left[\int_0^1 \frac{x - y}{(x + y)^3}\, \mathrm{d}y \right] \mathrm{d}x = \frac{1}{2} \qquad \text{and} \qquad \int_0^1 \left[\int_0^1 \frac{x - y}{(x + y)^3}\, \mathrm{d}x \right] \mathrm{d}y = -\frac{1}{2}\,.$$

Comment on these results.

11 For the tetrahedron V with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, evaluate the integral

$$\int_V x \, \mathrm{d}V.$$

Hence find the centre of volume $\frac{1}{V} \int_V \mathbf{x}\, \mathrm{d}V$.

12 A solid cone is bounded by the surface $\theta = \alpha$ in spherical polar coordinates and the surface $z = a$. Its mass density is $\rho_0 \cos\theta$. By evaluating a volume integral find the mass of the cone.

13 Express the integral

$$I = \int_0^\infty dx \int_0^1 dy \int_0^x dz \, x e^{-Ax/y - Bxy - Cyz}$$

in terms of the new variables $\alpha = x/y$, $\beta = xy$ and $\gamma = yz$. Hence show that

$$I = \frac{1}{2A(A + B)(A + B + C)}.$$

You may assume A, B and C are positive. [*Hint: Remember to calculate calculate the limits of the integral.*]

I would appreciate any comments and corrections from students and supervisors. Please e-mail `ejh1@cam.ac.uk`.