

This sheet contains a selection of (possibly more challenging) questions from recent Numbers&Sets examples sheets, set by I. B. Leader, W. T. Gowers or P. T. Johnstone. However, do not attempt these at the expense of the official sheets.

1. Let a and b be distinct positive integers, with $a < b$. Prove that every block of b consecutive positive integers contains two distinct numbers whose product is a multiple of ab . If a, b and c are distinct positive integers, with $a < b < c$, must every block of c consecutive positive integers contain three distinct numbers whose product is a multiple of abc ?
2. Let A be the sum of the digits of 4444^{4444} , and let B be the sum of the digits of A . What is the sum of the digits of B ?
3. Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?
4. Let x, y and z be positive integers satisfying $x^2 + y^2 + 1 = xyz$. Prove that $z = 3$.
5. Let $(x_n)_{n=1}^{\infty}$ be a real sequence with $x_n \rightarrow 0$. Prove that we may choose $(\epsilon_n)_{n=1}^{\infty}$, with each $\epsilon_n = \pm 1$, such that $\sum_{n=1}^{\infty} \epsilon_n x_n$ is convergent. If $(y_n)_{n=1}^{\infty}$ is another real sequence tending to 0, can we choose the ϵ_n so that $\sum_{n=1}^{\infty} \epsilon_n y_n$ is convergent as well?
6. Let f be a function from \mathbb{R}^2 to \mathbb{R} such that, for every x and y in \mathbb{R} , the functions $z \mapsto f(x, z)$ and $w \mapsto f(w, y)$ are polynomials. Prove that f is a polynomial in x and y . What happens if \mathbb{R} is replaced by \mathbb{Q} ? What if \mathbb{R} is replaced by \mathbb{Z}_p , for some prime p ?
7. Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that takes every value on every interval – in other words, such that for every $a < b$ and every c there is some x with $a < x < b$ and $f(x) = c$.
8. Let R be a rectangle which can be divided into smaller rectangles, each of which has at least one side of integer length. Prove that R has at least one side of integer length.
9. Given n points in the plane, not all collinear, show that it is possible to find a line containing exactly two of them.
10. Does there exist a cycle in \mathbb{Z}^3 (i.e., a path consisting of line segments joining neighbouring points which have integer coordinates, ending up where it started) such that none of the projections in the x, y and z directions contains a cycle?
11. Does there exist an uncountable family \mathcal{S} of subsets of \mathbb{N} such that for every (distinct) $A, B \in \mathcal{S}$, the intersection of A with B is finite?
12. Let \mathcal{S} be a collection of subsets of $\{1, \dots, n\}$ such that $|A \cap B|$ is even for all distinct A, B in \mathcal{S} . Prove that if $|A|$ is odd for all A in \mathcal{S} then $|\mathcal{S}| \leq n$. What if $|A|$ is even for all A in \mathcal{S} ?
13. (a) For $r \in \mathbb{N}$ and a set X , let $X^{(r)}$ be the set of subsets of X of size r . Show that if the members of $\mathbb{N}^{(r)}$ are each coloured either red or blue, then there is an infinite $X \subset \mathbb{N}$ such that all the members of $X^{(r)}$ have the same colour.
(b) This time, let the infinite subsets of \mathbb{N} be 2-coloured. Must there exist an infinite set $M \subset \mathbb{N}$ all of whose infinite subsets have the same colour?
14. Let $X = \{1, 2, 3, \dots, 2r + 1\}$. Give an explicit bijection $f : X^{(r)} \rightarrow X^{(r+1)}$ such that $A \subset f(A)$ for every $A \in X^{(r)}$.

15. Call a set X of positive integers a *multiplicative basis of order 2* if every positive integer n can be written as xy with x and y in X . Suppose that X is a multiplicative basis of order 2. Prove that there is a positive integer m that can be written as xy in at least 2008 different ways, all with x and y in X .
16. Let p be a prime other than 7. Show that every integer is a sum of two cubes mod p .
17. We have an infinite sequence of dons, and each is wearing a hat. The hats are red or blue, and each don can see every hat except his own. Simultaneously, each don has to shout out a guess as to the colour of his own hat. Can this be done in such a way that, *whatever* the distribution of hat colours, only finitely many dons guess incorrectly?
18. Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence $(x_n)_{n=1}^{\infty}$ such that, for each odd positive integer k , the series $\sum_{n=1}^{\infty} x_n^k$ converges when k belongs to S and diverges when k does not belong to S .
19. Let x_1, x_2, \dots be reals such that $\sum_{n=1}^{\infty} |x_n|$ is convergent. Show that if for every positive integer k we have $\sum_{n=1}^{\infty} x_{kn} = 0$ then $x_n = 0$ for all n . What happens if we drop the restriction that $\sum_{n=1}^{\infty} |x_n|$ is convergent?
- +20. Among a group of n dons, any two have exactly one mutual friend. Show that some don is friends with all the others.
- +21. Each of n elderly dons knows a piece of gossip not known to any of the others. They communicate by telephone, and in each call the two dons concerned reveal to each other all the information they know so far. What is the smallest number of calls that can be made in such a way that, at the end, all the dons know all the gossip?
- +22. Let $d \leq n$ be positive integers, with d even. How many subsets of $\{1, 2, \dots, n\}$ can we find such that any two have symmetric difference at most d ?
- +23. Let n be a fixed positive integer. Show that, for a sufficiently large prime p (i.e. for all but finitely many primes p), the equation $x^n + y^n = z^n$ has a solution in \mathbb{Z}_p with $x, y, z \neq 0$.