Paper 4, Section I

1E Numbers and Sets

(a) Find the smallest residue x which equals $28!\,13^{28} \pmod{31}$.

[*You may use any standard theorems provided you state them correctly.*]

(b) Find all integers x which satisfy the system of congruences

$$\begin{aligned} x &\equiv 1 \pmod{2}, \\ 2x &\equiv 1 \pmod{3}, \\ 2x &\equiv 4 \pmod{10}, \\ x &\equiv 10 \pmod{67}. \end{aligned}$$

Paper 4, Section I

2E Numbers and Sets

(a) Let r be a real root of the polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with integer coefficients a_i and leading coefficient 1. Show that if r is rational, then r is an integer.

(b) Write down a series for e. By considering $q!e$ for every natural number q, show that e is irrational.

Paper 4, Section II
5E Numbers and Sets

The Fibonacci numbers F_n are defined for all natural numbers n by the rules

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geqslant 3.$$

Prove by induction on k that, for any n,

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad \text{for all } k \geqslant 2.$$

Deduce that

$$F_{2n} = F_n(F_{n+1} + F_{n-1}) \qquad \text{for all } n \geqslant 2.$$

Put $L_1 = 1$ and $L_n = F_{n+1} + F_{n-1}$ for $n > 1$. Show that these (Lucas) numbers L_n satisfy

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geqslant 3.$$

Show also that, for all n, the greatest common divisor (F_n, F_{n+1}) is 1, and that the greatest common divisor (F_n, L_n) is at most 2.

Paper 4, Section II
6E Numbers and Sets

State and prove Fermat's Little Theorem.

Let p be an odd prime. If $p \neq 5$, show that p divides $10^n - 1$ for infinitely many natural numbers n.

Hence show that p divides infinitely many of the integers

$$5, \qquad 55, \qquad 555, \qquad 5555, \qquad \ldots\ .$$

Paper 4, Section II
7E Numbers and Sets

(a) Let A, B be finite non–empty sets, with $|A| = a,\ |B| = b$. Show that there are b^a mappings from A to B. How many of these are injective ?

(b) State the Inclusion–Exclusion principle.

(c) Prove that the number of surjective mappings from a set of size n onto a set of size k is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \qquad \text{for } n \geqslant k \geqslant 1 .$$

Deduce that

$$n! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n .$$

Paper 4, Section II
8E Numbers and Sets

What does it mean for a set to be countable ?

Show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not. Show also that the union of two countable sets is countable.

A subset A of $\mathbb{R}$ has the property that, given $\epsilon > 0$ and $x \in \mathbb{R}$, there exist reals a, b with $a \in A$ and $b \notin A$ with $|x-a| < \epsilon$ and $|x-b| < \epsilon$. Can A be countable ? Can A be uncountable ? Justify your answers.

A subset B of $\mathbb{R}$ has the property that given $b \in B$ there exists $\epsilon > 0$ such that if $0 < |b-x| < \epsilon$ for some $x \in \mathbb{R}$, then $x \notin B$. Is B countable ? Justify your answer.