4/I/1E Numbers and Sets

(i) Use Euclid's algorithm to find all pairs of integers x and y such that

$$7x + 18y = 1\,.$$

(ii) Show that, if n is odd, then $n^3 - n$ is divisible by 24.

4/I/2E Numbers and Sets

For integers k and n with $0 \leqslant k \leqslant n$, define $\binom{n}{k}$. Arguing from your definition, show that

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

for all integers k and n with $1 \leqslant k \leqslant n-1$.

Use induction on k to prove that

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$$

for all non-negative integers k and n.

4/II/5E Numbers and Sets

State and prove the Inclusion–Exclusion principle.

The keypad on a cash dispenser is broken. To withdraw money, a customer is required to key in a 4-digit number. However, the key numbered 0 will only function if either the immediately preceding two keypresses were both 1, or the very first key pressed was 2. Explaining your reasoning clearly, use the Inclusion–Exclusion Principle to find the number of 4-digit codes which can be entered.

4/II/6E Numbers and Sets

Stating carefully any results about countability you use, show that for any $d \geq 1$ the set $\mathbb{Z}[X_1, \ldots, X_d]$ of polynomials with integer coefficients in d variables is countable. By taking $d = 1$, deduce that there exist uncountably many transcendental numbers.

Show that there exists a sequence $x_1, x_2, \ldots$ of real numbers with the property that $f(x_1, \ldots, x_d) \neq 0$ for every $d \geq 1$ and for every non-zero polynomial $f \in \mathbb{Z}[X_1, \ldots, X_d]$.

[*You may assume without proof that* $\mathbb{R}$ *is uncountable.*]

4/II/7E Numbers and Sets

Let x_n $(n = 1, 2, \ldots)$ be real numbers.

What does it mean to say that the sequence $(x_n)_{n=1}^{\infty}$ converges?

What does it mean to say that the series $\sum_{n=1}^{\infty} x_n$ converges?

Show that if $\sum_{n=1}^{\infty} x_n$ is convergent, then $x_n \to 0$. Show that the converse can be false.

Sequences of positive real numbers x_n, y_n $(n \geq 1)$ are given, such that the inequality

$$y_{n+1} \leq y_n - \frac{1}{2}\min(x_n, y_n)$$

holds for all $n \geq 1$. Show that, if $\sum_{n=1}^{\infty} x_n$ diverges, then $y_n \to 0$.

4/II/8E Numbers and Sets

(i) Let p be a prime number, and let x and y be integers such that p divides xy. Show that at least one of x and y is divisible by p. Explain how this enables one to prove the Fundamental Theorem of Arithmetic.

[*Standard properties of highest common factors may be assumed without proof.*]

(ii) State and prove the Fermat-Euler Theorem.

Let $1/359$ have decimal expansion $0 \cdot a_1 a_2 \ldots$ with $a_n \in \{0, 1, \ldots, 9\}$. Use the fact that $60^2 \equiv 10 \pmod{359}$ to show that, for every n, $a_n = a_{n+179}$.