Mich. 2010 **NUMBERS AND SETS - EXAMPLES $4\frac{1}{2}$** **Gareth Taylor**

As an extension to N&S4, please attempt questions 2(i)-(iii), 4, 7, 9.
The rest of this sheet is for those who like this sort of thing.

1. Let A be the sum of the digits of 4444^{4444}, and let B be the sum of the digits of A. What is the sum of the digits of B?

2. Let (x_n) be a real sequence. Write $\sum$ to mean $\sum_{n=1}^{\infty}$.

   (i) Show that if $x_n \geqslant 0$ for all n and $\sum x_n$ converges then $\sum x_n^2$ converges. What if we drop the condition that $x_n \geqslant 0$?

   (ii) If $\sum x_n^2$ converges must $\sum x_n^3$ converge? If $\sum x_n^3$ converges must $\sum x_n^2$ converge?

   (iii) Show that if x_n is a decreasing sequence of positive numbers and $\sum x_n$ converges then $nx_n \to 0$ as $n \to \infty$.
   What if we drop the 'decreasing sequence' condition and just require $x_n \geqslant 0$?

   (iv) If $x_n \geqslant 0$ for all n and $\sum x_n$ converges, must $\sum \frac{\sqrt{x_n}}{n}$ converge?

   (v) (A bit harder.) If $\sum x_n$ converges, must $\sum \frac{|x_n|}{n}$ converge? Must $\sum \frac{x_n}{n}$ converge?

+3. Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence $(x_n)_{n=1}^{\infty}$ such that, for each odd positive integer k, the series $\sum_{n=1}^{\infty} x_n^k$ converges when k belongs to S and diverges when k does not belong to S.

4. *The Schröder-Bernstein theorem states that if there exist injections $f : A \to B$ and $g : B \to A$ between sets A and B, then there exists a bijection between A and B.*

   Use the Schröder-Bernstein theorem to show that there is a bijection between the open interval $(0, 1)$ and the closed interval $[0, 1]$. Now give an explicit bijection.

5. Use the Schröder-Bernstein theorem to show that there is a bijection between the set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ and the set $\mathbb{R}^{\mathbb{R}}$ of all functions $\mathbb{R} \to \mathbb{R}$.

6. A set A is said to be *Dedekind-infinite* if there exists an injective function $f : A \to A$ which is not surjective. Show that A is Dedekind-infinite if and only if there is an injective function $\mathbb{N} \to A$.

7. *(Add this final part to the official sheet 4 question 9.)* What happens if we replace 'discs' with 'figures of eight' (that is, things looking like '8', of any size and orientation)?

8. Let A be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$, and let $B = \{f \in A : f(n) = n$ for all but finitely many $n \in \mathbb{N}\}$. Show that A is uncountable, but that B is countable.

9. Let S be a *nested* collection of subsets of $\mathbb{N}$ – that is, for every $A, B \in S$ we have either $A \subset B$ or $B \subset A$. Can S be uncountable?

10. For each $x \in \mathbb{R}$ we are given an interval $I_x = [x - \delta_x, x + \delta_x]$ with $\delta_x \geqslant 0$. Moreover, for each $x, y \in \mathbb{R}$ with $y \in I_x$, we have $\delta_y < \delta_x$. Show that $\delta_x = 0$ for uncountably many x.

11. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval – in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.

+12. We have an infinite sequence of dons, and each is wearing a hat. The hats are red or blue, and each don can see every hat except his own. Simultaneously, each don has to shout out a guess as to the colour of his own hat. Can this be done in such a way that, whatever the distribution of hat colours, only finitely many dons guess incorrectly?

Please send any corrections or comments to me at glt1000@cam.ac.uk