

As an extension to N&S4, please attempt questions 2, 3, 6, 9, 10. For context, put 2 & 3 between the official sheet's 6 & 7, and put 10 before the official 13. The rest of this sheet is optional, and for those who like that sort of thing.

1. Let A be the sum of the digits of 4444^{4444} , and let B be the sum of the digits of A . What is the sum of the digits of B ?
2. Show that if $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, with all x_n positive, then $\sum_{n=1}^{\infty} x_n^2$ is also convergent. What happens if we do not insist that the x_n are positive?
3. If $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, must $\sum_{n=1}^{\infty} x_n^3$ be convergent?
4. Is there an enumeration of \mathbb{Q} as q_1, q_2, q_3, \dots such that $\sum (q_n - q_{n+1})^2$ converges?
- +5. Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence $(x_n)_{n=1}^{\infty}$ such that, for each odd positive integer k , the series $\sum_{n=1}^{\infty} x_n^k$ converges when k belongs to S and diverges when k does not belong to S .
6. Use the Schröder-Bernstein theorem to show that there is a bijection between the open interval $(0, 1)$ and the closed interval $[0, 1]$. Now give an explicit bijection. (*If the Schröder-Bernstein theorem wasn't covered in lectures, then take this question to be: give an explicit bijection between $(0, 1)$ and $[0, 1]$.*)
7. Use the Schröder-Bernstein theorem to show that there is a bijection between the set of all subsets of \mathbb{R} and the set of all functions $\mathbb{R} \rightarrow \mathbb{R}$.
8. A set A is said to be *Dedekind-infinite* if there exists an injective function $f : A \rightarrow A$ which is not surjective. Show that A is Dedekind-infinite if and only if there is an injective function $\mathbb{N} \rightarrow A$.
9. (*Add this final part to the official sheet 4 question 9.*) What happens if we replace 'discs' with 'figures of eight' (that is, things looking like '8', of any size and orientation)?
10. Let S be a *nested* collection of subsets of \mathbb{N} – that is, for every $A, B \in S$ we have either $A \subset B$ or $B \subset A$. Can S be uncountable?
11. Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that takes every value on every interval – in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.
- +12. We have an infinite sequence of dons, and each is wearing a hat. The hats are red or blue, and each don can see every hat except his own. Simultaneously, each don has to shout out a guess as to the colour of his own hat. Can this be done in such a way that, whatever the distribution of hat colours, only finitely many dons guess incorrectly?

Please send any corrections or comments to me at glt1000@cam.ac.uk