1. Give eight relations on the set $\mathbb{Z}$, one for each subset of {reflexive, symmetric, transitive}. That is: one obeying all of R, S, T; one obeying R and S but not T; etc.

2. Let $A_1, A_2, \ldots$ be sets such that for each n we have $A_1 \cap \ldots \cap A_n \neq \emptyset$. Does it follow that $A_1 \cap A_2 \cap \ldots \neq \emptyset$?

3. Let $f : X \to Y$ be a function. For $A \subseteq X$ let $f(A) = \{f(x) \mid x \in A\}$. For $B \subseteq Y$ let $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. Give proofs or counter-examples to the following claims:
   (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$; (b) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$;
   (c) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$; (d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

4. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

5. Use the Cantor-Bernstein theorem to show that there is a bijection between the open interval $(0, 1)$ and the closed interval $[0, 1]$. Now give an explicit bijection.

6. Show that the collection of all finite subsets of $\mathbb{N}$ is countable. What goes wrong if we try to use a diagonal argument to show that it is uncountable?

7. Show that there does not exist an uncountable family of pairwise disjoint discs in the plane. What if we replace 'discs' with 'circles'? What if we replace 'discs' with 'figures of eight' (that is, things looking like '8', of any size and orientation)?

8. A function $f : \mathbb{N} \to \mathbb{N}$ is called *increasing* if $f(n + 1) \geq f(n)$ for all n and *decreasing* if $f(n + 1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

9. Using the Cantor-Bernstein theorem, show that there is a bijection between the set of all subsets of $\mathbb{R}$ and the set of all functions $\mathbb{R} \to \mathbb{R}$.

10. Show that the set of finite strings of characters taken from a fixed finite alphabet is countable. Hence, by choosing a suitable alphabet, show that the set of all polynomials with integer coefficients is countable.

11. Find a bijection from $\mathbb{Q}$ to $\mathbb{Q} \setminus \{0\}$. Is there such a bijection that is order-preserving (i.e., $x < y$ implies $f(x) < f(y)$)?

12. Let S be a collection of subsets of $\mathbb{N}$ such that for every $A, B \in S$ we have $A \subseteq B$ or $B \subseteq A$. Can S be uncountable?

13. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval – in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.

+14. We have an infinite sequence of dons, and each is wearing a hat. The hats are red or blue, and each don can see every hat except his own. Simultaneously, each don has to shout out a guess as to the colour of his own hat. Can this be done in such a way that, whatever the distribution of hat colours, only finitely many dons guess incorrectly?

Please send any corrections or comments to me at glt1000@cam.ac.uk