Paper 2, Section I

4G Metric and Topological Spaces

(i) Let $t > 0$. For $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y') \in \mathbb{R}^2$, let

$$d(\mathbf{x}, \mathbf{x}') = |x' - x| + t|y' - y|,$$

$$\delta(\mathbf{x}, \mathbf{x}') = \sqrt{(x' - x)^2 + (y' - y)^2}.$$

(δ is the usual Euclidean metric on $\mathbb{R}^2$.) Show that d is a metric on $\mathbb{R}^2$ and that the two metrics d, δ give rise to the same topology on $\mathbb{R}^2$.

(ii) Give an example of a topology on $\mathbb{R}^2$, different from the one in (i), whose induced topology (subspace topology) on the x-axis is the usual topology (the one defined by the metric $d(x, x') = |x' - x|$). Justify your answer.

Paper 3, Section I

3G Metric and Topological Spaces

Let X, Y be topological spaces, and suppose Y is Hausdorff.

(i) Let $f, g : X \to Y$ be two continuous maps. Show that the set

$$E(f, g) := \{x \in X \mid f(x) = g(x)\} \subset X$$

is a closed subset of X.

(ii) Let W be a dense subset of X. Show that a continuous map $f : X \to Y$ is determined by its restriction $f|_W$ to W.

Paper 1, Section II

12G Metric and Topological Spaces

Let X be a metric space with the distance function $d : X \times X \to \mathbb{R}$. For a subset Y of X, its *diameter* is defined as $\delta(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$.

Show that, if X is compact and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of X, then there exists an $\epsilon > 0$ such that every subset $Y \subset X$ with $\delta(Y) < \epsilon$ is contained in some U_λ.

Paper 4, Section II

13G Metric and Topological Spaces

Let X, Y be topological spaces and $X \times Y$ their product set. Let $p_Y : X \times Y \to Y$ be the projection map.

(i) Define the product topology on $X \times Y$. Prove that if a subset $Z \subset X \times Y$ is open then $p_Y(Z)$ is open in Y.

(ii) Give an example of X, Y and a closed set $Z \subset X \times Y$ such that $p_Y(Z)$ is not closed.

(iii) When X is compact, show that if a subset $Z \subset X \times Y$ is closed then $p_Y(Z)$ is closed.