Paper 2, Section I

4F Metric and Topological Spaces

Explain what is meant by a Hausdorff (topological) space, and prove that every compact subset of a Hausdorff space is closed.

Let X be an uncountable set, and consider the topology $\mathcal{T}$ on X defined by

$$U \in \mathcal{T} \Leftrightarrow \text{ either } U = \emptyset \text{ or } X \setminus U \text{ is countable.}$$

Is $(X, \mathcal{T})$ Hausdorff? Is every compact subset of X closed? Justify your answers.

Paper 3, Section I

4F Metric and Topological Spaces

Are the following statements true or false? Give brief justifications for your answers.

(i) If X is a connected open subset of $\mathbb{R}^n$ for some n, then X is path-connected.

(ii) A cartesian product of two connected spaces is connected.

(iii) If X is a Hausdorff space and the only connected subsets of X are singletons $\{x\}$, then X is discrete.

Paper 1, Section II

12F Metric and Topological Spaces

Given a function $f : X \to Y$ between metric spaces, we write Γ_f for the subset $\{(x, f(x)) \mid x \in X\}$ of $X \times Y$.

(i) If f is continuous, show that Γ_f is closed in $X \times Y$.

(ii) If Y is compact and Γ_f is closed in $X \times Y$, show that f is continuous.

(iii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that Γ_f is closed but f is not continuous.

Paper 4, Section II

14F Metric and Topological Spaces

A nonempty subset A of a topological space X is called *irreducible* if, whenever we have open sets U and V such that $U \cap A$ and $V \cap A$ are nonempty, then we also have $U \cap V \cap A \neq \emptyset$. Show that the closure of an irreducible set is irreducible, and deduce that the closure of any singleton set $\{x\}$ is irreducible.

X is said to be a *sober* topological space if, for any irreducible closed $A \subseteq X$, there is a unique $x \in X$ such that $A = \overline{\{x\}}$. Show that any Hausdorff space is sober, but that an infinite set with the cofinite topology is not sober.

Given an arbitrary topological space $(X, \mathcal{T})$, let $\widehat{X}$ denote the set of all irreducible closed subsets of X, and for each $U \in \mathcal{T}$ let

$$\widehat{U} = \{A \in \widehat{X} \mid U \cap A \neq \emptyset\} \ .$$

Show that the sets $\{\widehat{U} \mid U \in \mathcal{T}\}$ form a topology $\widehat{\mathcal{T}}$ on $\widehat{X}$, and that the mapping $U \mapsto \widehat{U}$ is a bijection from $\mathcal{T}$ to $\widehat{\mathcal{T}}$. Deduce that $(\widehat{X}, \widehat{\mathcal{T}})$ is sober. [*Hint: consider the complement of an irreducible closed subset of* $\widehat{X}$.]