Paper 1, Section I

1G Linear Algebra

(i) State the rank-nullity theorem for a linear map between finite-dimensional vector spaces.

(ii) Show that a linear transformation $f : V \to V$ of a finite-dimensional vector space V is bijective if it is injective or surjective.

(iii) Let V be the $\mathbb{R}$-vector space $\mathbb{R}[X]$ of all polynomials in X with coefficients in $\mathbb{R}$. Give an example of a linear transformation $f : V \to V$ which is surjective but not bijective.

Paper 2, Section I

1G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space with an inner product. Let W be an m-dimensional subspace of V and $W^{\perp}$ its orthogonal complement, so that every element $v \in V$ can be uniquely written as $v = w + w'$ for $w \in W$ and $w' \in W^{\perp}$.

The *reflection map* with respect to W is defined as the linear map

$$f_W : V \ni w + w' \longmapsto w - w' \in V.$$

Show that f_W is an orthogonal transformation with respect to the inner product, and find its determinant.

Paper 4, Section I

1G Linear Algebra

(i) Let V be a vector space over a field F, and W_1, W_2 subspaces of V. Define the subset $W_1 + W_2$ of V, and show that $W_1 + W_2$ and $W_1 \cap W_2$ are subspaces of V.

(ii) When W_1, W_2 are finite-dimensional, state a formula for $\dim(W_1 + W_2)$ in terms of $\dim W_1$, $\dim W_2$ and $\dim(W_1 \cap W_2)$.

(iii) Let V be the $\mathbb{R}$-vector space of all $n \times n$ matrices over $\mathbb{R}$. Let S be the subspace of all symmetric matrices and T the subspace of all upper triangular matrices (the matrices (a_{ij}) such that $a_{ij} = 0$ whenever $i > j$). Find $\dim S$, $\dim T$, $\dim(S \cap T)$ and $\dim(S + T)$. Briefly justify your answer.

Paper 1, Section II

9G Linear Algebra

Let V, W be finite-dimensional vector spaces over a field F and $f : V \to W$ a linear map.

(i) Show that f is injective if and only if the image of every linearly independent subset of V is linearly independent in W.

(ii) Define the dual space V^* of V and the dual map $f^* : W^* \to V^*$.

(iii) Show that f is surjective if and only if the image under f^* of every linearly independent subset of W^* is linearly independent in V^*.

Paper 2, Section II

10G Linear Algebra

Let n be a positive integer, and let V be a $\mathbb{C}$-vector space of complex-valued functions on $\mathbb{R}$, generated by the set $\{ \cos kx,\ \sin kx;\ k = 0, 1, \ldots, n-1 \}$.

(i) Let $\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$ for $f, g \in V$. Show that this is a positive definite Hermitian form on V.

(ii) Let $\Delta(f) = \dfrac{d^2}{dx^2} f(x)$. Show that Δ is a self-adjoint linear transformation of V with respect to the form defined in (i).

(iii) Find an orthonormal basis of V with respect to the form defined in (i), which consists of eigenvectors of Δ.

Paper 3, Section II

10G Linear Algebra

(i) Let A be an $n \times n$ complex matrix and $f(X)$ a polynomial with complex coefficients. By considering the Jordan normal form of A or otherwise, show that if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$. Write B as $B = f(A)$ for a polynomial f with $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and find the eigenvalues of B.

[Hint: compute the powers of A.]

Paper 4, Section II

10G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space and $f, g : V \to V$ linear transformations. Suppose f is invertible and diagonalisable, and $f \circ g = t \cdot (g \circ f)$ for some real number $t > 1$.

(i) Show that g is nilpotent, i.e. some positive power of g is 0.

(ii) Suppose that there is a non-zero vector $v \in V$ with $f(v) = v$ and $g^{n-1}(v) \neq 0$. Determine the diagonal form of f.