Paper 1, Section I

1F Linear Algebra

Suppose that V is the complex vector space of polynomials of degree at most $n-1$ in the variable z. Find the Jordan normal form for each of the linear transformations $\frac{d}{dz}$ and $z\frac{d}{dz}$ acting on V.

Paper 2, Section I

1F Linear Algebra

Suppose that ϕ is an endomorphism of a finite-dimensional complex vector space.

(i) Show that if λ is an eigenvalue of ϕ, then λ^2 is an eigenvalue of ϕ^2.

(ii) Show conversely that if μ is an eigenvalue of ϕ^2, then there is an eigenvalue λ of ϕ with $\lambda^2 = \mu$.

Paper 4, Section I

1F Linear Algebra

Define the notion of an inner product on a finite-dimensional real vector space V, and the notion of a self-adjoint linear map $\alpha : V \to V$.

Suppose that V is the space of real polynomials of degree at most n in a variable t. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

is an inner product on V, and that the map $\alpha : V \to V$:

$$\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)$$

is self-adjoint.

Paper 1, Section II

9F Linear Algebra

Let V denote the vector space of $n \times n$ real matrices.

(1) Show that if $\psi(A, B) = \operatorname{tr}(AB^T)$, then ψ is a positive-definite symmetric bilinear form on V.

(2) Show that if $q(A) = \operatorname{tr}(A^2)$, then q is a quadratic form on V. Find its rank and signature.

[Hint: Consider symmetric and skew-symmetric matrices.]

Paper 2, Section II

10F Linear Algebra

(i) Show that two $n \times n$ complex matrices A, B are similar (i.e. there exists invertible P with $A = P^{-1}BP$) if and only if they represent the same linear map $\mathbb{C}^n \to \mathbb{C}^n$ with respect to different bases.

(ii) Explain the notion of Jordan normal form of a square complex matrix.

(iii) Show that any square complex matrix A is similar to its transpose.

(iv) If A is invertible, describe the Jordan normal form of A^{-1} in terms of that of A.

Justify your answers.

Paper 3, Section II

10F Linear Algebra

Suppose that V is a finite-dimensional vector space over $\mathbb{C}$, and that $\alpha : V \to V$ is a $\mathbb{C}$-linear map such that $\alpha^n = 1$ for some $n > 1$. Show that if V_1 is a subspace of V such that $\alpha(V_1) \subset V_1$, then there is a subspace V_2 of V such that $V = V_1 \oplus V_2$ and $\alpha(V_2) \subset V_2$.

[Hint: Show, for example by picking bases, that there is a linear map $\pi : V \to V_1$ with $\pi(x) = x$ for all $x \in V_1$. Then consider $\rho : V \to V_1$ with $\rho(y) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha^i \pi \alpha^{-i}(y)$.]

Paper 4, Section II

10F Linear Algebra

(i) Show that the group $O_n(\mathbb{R})$ of orthogonal $n \times n$ real matrices has a normal subgroup $SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) \mid \det A = 1\}$.

(ii) Show that $O_n(\mathbb{R}) = SO_n(\mathbb{R}) \times \{\pm I_n\}$ if and only if n is odd.

(iii) Show that if n is even, then $O_n(\mathbb{R})$ is not the direct product of $SO_n(\mathbb{R})$ with any normal subgroup.

[*You may assume that the only elements of* $O_n(\mathbb{R})$ *that commute with all elements of* $O_n(\mathbb{R})$ *are* $\pm I_n$.]