Paper 1, Section I

1G Linear Algebra

(1) Let V be a finite-dimensional vector space and let $T : V \to V$ be a non-zero endomorphism of V. If $\ker(T) = \operatorname{im}(T)$ show that the dimension of V is an even integer. Find the minimal polynomial of T. [*You may assume the rank-nullity theorem.*]

(2) Let A_i, $1 \leqslant i \leqslant 3$, be non-zero subspaces of a vector space V with the property that

$$V = A_1 \oplus A_2 = A_2 \oplus A_3 = A_1 \oplus A_3.$$

Show that there is a 2-dimensional subspace $W \subset V$ for which all the $W \cap A_i$ are one-dimensional.

Paper 2, Section I

1G Linear Algebra

Let V denote the vector space of polynomials $f(x,y)$ in two variables of total degree at most n. Find the dimension of V.

If $S : V \to V$ is defined by

$$(Sf)(x,y) = x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2},$$

find the kernel of S and the image of S. Compute the trace of S for each n with $1 \leqslant n \leqslant 4$.

Paper 4, Section I

1G Linear Algebra

Show that every endomorphism of a finite-dimensional vector space satisfies some polynomial, and define the *minimal polynomial* of such an endomorphism.

Give a linear transformation of an eight-dimensional complex vector space which has minimal polynomial $x^2(x-1)^3$.

Paper 1, Section II

9G Linear Algebra

Define the *dual* of a vector space V. State and prove a formula for its dimension.

Let V be the vector space of real polynomials of degree at most n. If $\{a_0, \ldots, a_n\}$ are distinct real numbers, prove that there are unique real numbers $\{\lambda_0, \ldots, \lambda_n\}$ with

$$\frac{dp}{dx}(0) \;=\; \sum_{j=0}^{n} \lambda_j p(a_j)$$

for every $p(x) \in V$.

Paper 2, Section II

10G Linear Algebra

Let V be a finite-dimensional vector space and let $T : V \to V$ be an endomorphism of V. Show that there is a positive integer l such that $V = \ker(T^l) \oplus \operatorname{im}(T^l)$. Hence, or otherwise, show that if T has zero determinant there is some non-zero endomorphism S with $TS = 0 = ST$.

Suppose T_1 and T_2 are endomorphisms of V for which $T_i^2 = T_i$, $i = 1, 2$. Show that T_1 is similar to T_2 if and only if they have the same rank.

Paper 3, Section II

10G Linear Algebra

For each of the following, provide a proof or counterexample.

(1) If A, B are complex $n \times n$ matrices and $AB = BA$, then A and B have a common eigenvector.

(2) If A, B are complex $n \times n$ matrices and $AB = BA$, then A and B have a common eigenvalue.

(3) If A, B are complex $n \times n$ matrices and $(AB)^n = 0$ then $(BA)^n = 0$.

(4) If $T : V \to V$ is an endomorphism of a finite-dimensional vector space V and λ is an eigenvalue of T, then the dimension of $\{v \in V \mid (T - \lambda I)v = 0\}$ equals the multiplicity of λ as a root of the minimal polynomial of T.

(5) If $T : V \to V$ is an endomorphism of a finite-dimensional complex vector space V, λ is an eigenvalue of T, and $W_i = \{v \in V \mid (T - \lambda I)^i(v) = 0\}$, then $W_c = W_{c+1}$ where c is the multiplicity of λ as a root of the minimal polynomial of T.

Paper 4, Section II

10G Linear Algebra

What does it mean to say two real symmetric bilinear forms A and B on a vector space V are *congruent* ?

State and prove Sylvester's law of inertia, and deduce that the rank and signature determine the congruence class of a real symmetric bilinear form. [*You may use without proof a result on diagonalisability of real symmetric matrices, provided it is clearly stated.*]

How many congruence classes of symmetric bilinear forms on a real n-dimensional vector space are there? Such a form ψ defines a family of subsets $\{x \in \mathbb{R}^n \mid \psi(x,x) = t\}$, for $t \in \mathbb{R}$. For how many of the congruence classes are these associated subsets all bounded subsets of $\mathbb{R}^n$? Is the quadric surface

$$\{3x^2 + 6y^2 + 5z^2 + 4xy + 2xz + 8yz = 1\}$$

a bounded or unbounded subset of $\mathbb{R}^3$? Justify your answers.