

## GRM Review Problems

*Exercise 0.1.* Show that if  $H$  and  $K$  are subgroups of a group  $G$ , then  $H \cap K$  is also a subgroup of  $G$ . Show also that, if  $H$  and  $K$  have orders  $p$  and  $q$ , respectively, where  $p$  and  $q$  are coprime, then  $H \cap K$  contains only the identity element  $e$  of  $G$ .

*Exercise 0.2.* Suppose  $G$  is group in which every element other than the identity has order 2. By evaluating  $x(xy)^2y$  in two ways, show that  $xy = yx$  for all  $x, y \in G$ . If the identity  $e$ ,  $x$  and  $y$  are all distinct, show that the set  $\{e, x, y, xy\}$  is a subgroup of  $G$  of order exactly 4.

Use Lagrange's theorem to show that any group of order  $2p$ , where  $p$  is an odd prime, must contain an element of order  $p$ .

*Exercise 0.3.* Let  $C_n$  be the cyclic group with  $n$  elements and  $D_{2n}$  the dihedral group with  $2n$  elements (i.e., the group of symmetries of the regular  $n$ -gon<sup>1</sup>). If  $n$  is odd and  $f : D_{2n} \rightarrow C_n$  is a homomorphism, show that  $f(x) = e$  for all  $x \in D_{2n}$ . What can we say if  $n$  is even?

Find all the homomorphisms from the cyclic group  $C_n$  of order  $n$  generated by  $a$ , say, to  $C_m$  the cyclic group generated by  $b$ , say. If  $n = m$ , show that there are  $\phi(n)$  isomorphisms, where  $\phi(n)$  is the number of integers between 0 and  $n - 1$  coprime to  $n$  (Euler's totient function).

*Exercise 0.4.* The dihedral group  $D_{2n}$  is the full symmetry group of a regular plane  $n$ -gon. Show that, if the integer  $m$  divides  $2n$ , then  $D_{2n}$  has a subgroup of order  $m$ .

If  $n \geq m \geq 3$ , show that  $D_{2m}$  is isomorphic to a subgroup of  $D_{2n}$  if and only if  $m$  divides  $n$ .

*Exercise 0.5.* Let  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  be the additive groups of integers, rational and real numbers respectively. Show that every element of the quotient group  $\mathbb{Q}/\mathbb{Z}$  has finite order. Show that every element of the quotient group  $\mathbb{R}/\mathbb{Q}$  (apart from the identity) has infinite order. Show that some elements of  $\mathbb{R}/\mathbb{Z}$  have infinite order and some non-identity elements do not.

*Exercise 0.6.* The group of  $2 \times 2$  real non-singular matrices is the General Linear Group  $GL(2, \mathbb{R})$ ; the subset of  $GL(2, \mathbb{R})$  consisting of matrices of determinant 1 is called the Special Linear Group  $SL(2, \mathbb{R})$ . Show that  $SL(2, \mathbb{R})$  is, indeed, a subgroup of  $GL(2, \mathbb{R})$  and that it is, in fact, normal. Show that the quotient group  $GL(2, \mathbb{R})/SL(2, \mathbb{R})$  is isomorphic to the multiplicative group of non-zero real numbers. [The neatest way to do this question is to reflect on the isomorphism theorem (Theorem ??).]

*Exercise 0.7.* Let  $G$  and  $H$  be groups and  $\phi : G \rightarrow H$  a homomorphism with kernel  $K$ . Show that, if  $K = \{e, a\}$ , then  $x^{-1}ax = a$  for all  $x \in G$ .

Show that:-

(i) There is a homomorphism from  $O(3)$ , the orthogonal group of  $3 \times 3$  real matrices, onto a group of order 2 with kernel the special orthogonal group  $SO(3)$ .

(ii) There is a homomorphism from  $S_3$  the symmetry group on 3 elements to a group of order 2 with a kernel of order 3.

(iii) There is a homomorphism from  $O(3)$  onto  $SO(3)$  with kernel of order 2.

(iv) There is no homomorphism from  $S_3$  to a group of order 3 with a kernel of order 2.

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<sup>1</sup>Observe my inability to keep to a consistent choice between  $D_n$  and  $D_{2n}$ .

*Exercise 0.8.* Let  $G$  be a finite group and  $X$  the set of its subgroups. Show that  $g(L) = gLg^{-1}$  [ $g \in G, L \in X$ ] defines an action of  $G$  on  $X$ . If  $H$  is a proper subgroup of  $G$  show that the orbit of  $H$  has at most  $|G|/|H|$  elements and, by considering overlapping, or otherwise, show that there is an element of  $G$  which does not belong to any conjugate of  $H$ .

*Exercise 0.9.* If  $G$  is a group, we call an isomorphism  $\alpha : G \rightarrow G$  an *automorphism*. Show that the automorphisms of  $G$  form a group under composition.

Consider  $\mathbb{Q}$  (the rationals) as an additive group. Show that, if  $r$  and  $s$  are non-zero rationals, there is a unique automorphism  $\alpha$  with  $\alpha(r) = s$ . Deduce that the group of automorphisms of  $\mathbb{Q}$  is isomorphic to the multiplicative group of  $\mathbb{Q} \setminus \{0\}$ .

*Exercise 0.10.* You are skiing on the border of Syldavia. By mistake you cross into Borduria and are arrested. The border guard turns out to be an old Trinity man and agrees to let you go provided that you prove you are indeed a mathematician by classifying all groups of order 10. Do so.

*Exercise 0.11.* What is the largest possible order of an element in  $S_5$ ?

What is the largest possible order of an element in  $S_9$ ?

What is the largest possible order of an element in  $S_{16}$ ? [You may need to run through several possibilities.]

Show that every element in  $S_{10}$  of order 14 is odd.

*Exercise 0.12.* Show that any subgroup of  $S_n$  which is not contained in  $A_n$  contains an equal number of odd and even elements.

*Exercise 0.13.* The *cycle type* of an element  $\sigma$  of the symmetric group  $S_n$  is defined to be the collection of lengths of disjoint cycles that form  $\sigma$ . (For example  $(1754)(268)(3)(9) \in S_9$  has cycle type 4, 3, 1, 1.)

(i) Show that  $\sigma_1$  and  $\sigma_2$  in  $S_n$  have the same cycle type if and only if there exists a  $\tau \in S_n$  such that  $\sigma_1 = \tau^{-1}\sigma_2\tau$ .

(ii) Find the number of elements of each cycle type in  $S_5$ . Which of them belong to  $A_5$ ?

*Exercise 0.14.* (i) Show that  $S_n$  is generated by transpositions of the form  $(1j)$  with  $2 \leq j \leq n$ .

(ii) Show that  $S_n$  is generated by transpositions of the form  $(j-1j)$  with  $2 \leq j \leq n$ .

(iii) Show that  $S_n$  is generated by the two elements  $(12)$  and  $(123\dots n)$ .

(iv) For which values of  $n$  is  $S_n$  generated by a single element? Prove your answer.

(v) Calculate the product  $(12)(13)$  in  $S_n$  for  $n \geq 3$ . Calculate the product  $(123)(124)$  in  $S_n$  for  $n \geq 4$ . Show that, if  $n \geq 3$ ,  $A_n$  is generated by the set of all cycles of length 3 in  $S_n$ . What happens if  $n = 2$  or  $n = 1$ ?

*Exercise 0.15.* (i) Show that, if  $n \geq 5$ , then  $S_n$  is generated by 4-cycles (that is to say, cycles of length 4). Can the identity can be written as the product of an odd number of 4-cycles?

(ii) Let  $n \geq 3$  and let  $X$  be the subset of  $S_n$  consisting of those  $\sigma$  with  $\sigma 1 = 2$ . Show that  $S_n$  is generated by  $X$ . Can we define a function  $\Omega : S_n \rightarrow \{-1, 1\}$  by taking  $\Omega(\sigma) = (-1)^n$  if  $\sigma$  is the product of  $n$  elements of  $X$  and their inverses?

(iii) If  $G$  is an Abelian group and  $T : S_n \rightarrow G$  is a homomorphism, what can you say about the image  $T(S_n)$ ?