**Paper 2, Section I**

**2F Groups, Rings and Modules**

Show that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$, is not isomorphic to the symmetry group $D_8$ of the square.

**Paper 3, Section I**

**1F Groups, Rings and Modules**

Suppose that $A$ is an integral domain containing a field $K$ and that $A$ is finite-dimensional as a $K$-vector space. Prove that $A$ is a field.

**Paper 4, Section I**

**2F Groups, Rings and Modules**

A ring $R$ satisfies the descending chain condition (DCC) on ideals if, for every sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of ideals in $R$, there exists $n$ with $I_n = I_{n+1} = I_{n+2} = \dots$. Show that $\mathbb{Z}$ does not satisfy the DCC on ideals.

**Paper 1, Section II**

**10F Groups, Rings and Modules**

(i) Suppose that $G$ is a finite group of order $p^n r$, where $p$ is prime and does not divide $r$. Prove the first Sylow theorem, that $G$ has at least one subgroup of order $p^n$, and state the remaining Sylow theorems without proof.

(ii) Suppose that $p, q$ are distinct primes. Show that there is no simple group of order $pq$.

**Paper 2, Section II**

**11F Groups, Rings and Modules**

Define the notion of a Euclidean domain and show that $\mathbb{Z}[i]$ is Euclidean.

Is $4 + i$ prime in $\mathbb{Z}[i]$?

**Paper 3, Section II**

**11F Groups, Rings and Modules**

Suppose that $A$ is a matrix over $\mathbb{Z}$. What does it mean to say that $A$ can be brought to Smith normal form?

Show that the structure theorem for finitely generated modules over $\mathbb{Z}$ (which you should state) follows from the existence of Smith normal forms for matrices over $\mathbb{Z}$.

Bring the matrix $\begin{pmatrix} -4 & -6 \\ 2 & 2 \end{pmatrix}$ to Smith normal form.

Suppose that $M$ is the $\mathbb{Z}$-module with generators $e_1, e_2$, subject to the relations

$$-4e_1 + 2e_2 = -6e_1 + 2e_2 = 0\,.$$

Describe $M$ in terms of the structure theorem.

**Paper 4, Section II**

**11F Groups, Rings and Modules**

State and prove the Hilbert Basis Theorem.

Is every ring Noetherian? Justify your answer.