

Paper 2, Section I**2H Groups Rings and Modules**

Give the definition of conjugacy classes in a group G . How many conjugacy classes are there in the symmetric group S_4 on four letters? Briefly justify your answer.

Paper 3, Section I**1H Groups Rings and Modules**

Let A be the ring of integers \mathbb{Z} or the polynomial ring $\mathbb{C}[X]$. In each case, give an example of an ideal I of A such that the quotient ring $R = A/I$ has a non-trivial idempotent (an element $x \in R$ with $x \neq 0, 1$ and $x^2 = x$) and a non-trivial nilpotent element (an element $x \in R$ with $x \neq 0$ and $x^n = 0$ for some positive integer n). Exhibit these elements and justify your answer.

Paper 4, Section I**2H Groups Rings and Modules**

Let M be a free \mathbb{Z} -module generated by e_1 and e_2 . Let a, b be two non-zero integers, and N be the submodule of M generated by $ae_1 + be_2$. Prove that the quotient module M/N is free if and only if a, b are coprime.

Paper 1, Section II**10H Groups Rings and Modules**

Prove that the kernel of a group homomorphism $f : G \rightarrow H$ is a normal subgroup of the group G .

Show that the dihedral group D_8 of order 8 has a non-normal subgroup of order 2. Conclude that, for a group G , a normal subgroup of a normal subgroup of G is not necessarily a normal subgroup of G .

Paper 2, Section II
11H Groups Rings and Modules

For ideals I, J of a ring R , their *product* IJ is defined as the ideal of R generated by the elements of the form xy where $x \in I$ and $y \in J$.

- (1) Prove that, if a prime ideal P of R contains IJ , then P contains either I or J .
- (2) Give an example of R, I and J such that the two ideals IJ and $I \cap J$ are different from each other.
- (3) Prove that there is a natural bijection between the prime ideals of R/IJ and the prime ideals of $R/(I \cap J)$.

Paper 3, Section II
11H Groups Rings and Modules

Let R be an integral domain and R^\times its group of units. An element of $S = R \setminus (R^\times \cup \{0\})$ is *irreducible* if it is not a product of two elements in S . When R is Noetherian, show that every element of S is a product of finitely many irreducible elements of S .

Paper 4, Section II
11H Groups Rings and Modules

Let $V = (\mathbb{Z}/3\mathbb{Z})^2$, a 2-dimensional vector space over the field $\mathbb{Z}/3\mathbb{Z}$, and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V$.

- (1) List all 1-dimensional subspaces of V in terms of e_1, e_2 . (For example, there is a subspace $\langle e_1 \rangle$ generated by e_1 .)
- (2) Consider the action of the matrix group

$$G = GL_2(\mathbb{Z}/3\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/3\mathbb{Z}, ad - bc \neq 0 \right\}$$

on the finite set X of all 1-dimensional subspaces of V . Describe the stabiliser group K of $\langle e_1 \rangle \in X$. What is the order of K ? What is the order of G ?

- (3) Let $H \subset G$ be the subgroup of all elements of G which act trivially on X . Describe H , and prove that G/H is isomorphic to S_4 , the symmetric group on four letters.