Paper 2, Section I

2F Groups, Rings and Modules

State Sylow's theorems. Use them to show that a group of order 56 must have either a normal subgroup of order 7 or a normal subgroup of order 8.

Paper 3, Section I

1F Groups, Rings and Modules

Let F be a field. Show that the polynomial ring $F[X]$ is a principal ideal domain. Give, with justification, an example of an ideal in $F[X,Y]$ which is not principal.

Paper 4, Section I

2F Groups, Rings and Modules

Let M be a module over an integral domain R. An element $m \in M$ is said to be torsion if there exists a nonzero $r \in R$ with $rm = 0$; M is said to be torsion-free if its only torsion element is 0. Show that there exists a unique submodule N of M such that (a) all elements of N are torsion and (b) the quotient module M/N is torsion-free.

Paper 1, Section II

10F Groups, Rings and Modules

Prove that a principal ideal domain is a unique factorization domain.

Give, with justification, an example of an element of $\mathbb{Z}[\sqrt{-3}]$ which does not have a unique factorization as a product of irreducibles. Show how $\mathbb{Z}[\sqrt{-3}]$ may be embedded as a subring of index 2 in a ring R (that is, such that the additive quotient group $R/\mathbb{Z}[\sqrt{-3}]$ has order 2) which is a principal ideal domain. [*You should explain why R is a principal ideal domain, but detailed proofs are not required.*]

Paper 2, Section II

11F Groups, Rings and Modules

Define the centre of a group, and prove that a group of prime-power order has a nontrivial centre. Show also that if the quotient group $G/Z(G)$ is cyclic, where $Z(G)$ is the centre of G, then it is trivial. Deduce that a non-abelian group of order p^3, where p is prime, has centre of order p.

Let F be the field of p elements, and let G be the group of 3×3 matrices over F of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} .$$

Identify the centre of G.

Paper 3, Section II

11F Groups, Rings and Modules

Let S be a multiplicatively closed subset of a ring R, and let I be an ideal of R which is maximal among ideals disjoint from S. Show that I is prime.

If R is an integral domain, explain briefly how one may construct a field F together with an injective ring homomorphism $R \to F$.

Deduce that if R is an arbitrary ring, I an ideal of R, and S a multiplicatively closed subset disjoint from I, then there exists a ring homomorphism $f : R \to F$, where F is a field, such that $f(x) = 0$ for all $x \in I$ and $f(y) \neq 0$ for all $y \in S$.

[*You may assume that if T is a multiplicatively closed subset of a ring, and $0 \notin T$, then there exists an ideal which is maximal among ideals disjoint from T.*]

Paper 4, Section II

11F Groups, Rings and Modules

Let R be a principal ideal domain. Prove that any submodule of a finitely-generated free module over R is free.

An R-module P is said to be projective if, whenever we have module homomorphisms $f : M \to N$ and $g : P \to N$ with f surjective, there exists a homomorphism $h \colon P \to M$ with $f \circ h = g$. Show that any free module (over an arbitrary ring) is projective. Show also that a finitely-generated projective module over a principal ideal domain is free.