Mich. 2011 **GROUPS PRELIMINARY SHEET** **Gareth Taylor**

1. *Associativity continues*

   Let $(G,*)$ be a group. Then for all $a,b,c \in G$, we have $a*(b*c) = (a*b)*c$. This allows us to write $a*b*c$ without any ambiguity, as both ways of bracketing the terms give equal expressions.

   How many different ways of bracketing $a*b*c*d$ are there? Using the above rule of associativity for three terms, show that all of these ways of bracketing four terms give equal expressions.

2. *In general,* $(ab)^n \neq a^n b^n$

   (a) Let G be a group, and let $a,b \in G$. Show that if $ab = ba$ then $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$. Show also that if $(ab)^n = a^n b^n$ for all n, then $ab = ba$. (Note that 'for all n' is needlessly strong!)

   (b) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $AB \neq BA$, and that $(AB)^2 \neq A^2B^2$. More generally, for $n \in \mathbb{N}$, find $(AB)^n$ and A^nB^n.

   (c) (Optional.) Find 2×2 matrices C, D such that $(CD)^n = C^nD^n$ for all n, but $CD \neq DC$.

3. *Commutativity* $\not\Rightarrow$ *associativity*

   Let $S = \{0,1,2,3,4,5\}$, and define the operation $*$ on S by $x * y = |x - y|$.

   Show that: this operation is closed; there is an identity element; every element has an inverse; the operation is commutative; the operation is not associative.

4. *Cayley tables*

   A *Latin square* is an array of symbols such that each row and column contains the same set of symbols, and no row or column contains a duplicated entry. (So, for example, a correctly completed Sudoku grid forms a Latin square.)

   (a) Explain why the multiplication table (officially, 'Cayley table') of a group forms a Latin square. Explain how one can read off from the table what the identity is, what the inverse of an element is, and whether the group is commutative. (Associativity is not obvious from the table.)

   (b) Below are incomplete Cayley tables for two groups, each of whose identity elements is called e. Complete the tables. (You will need to use more group properties than just 'Latin square'. Also, please give some reasoning rather than just the finished table.)

$*$	e	x	y	z
e				
x		z		
y				
z				

$*$	e	p	q	r	s	t
e						
p						
q						
r			p			
s		r		t		
t				e		

   Assume the convention used is 'left $*$ top', so that, for example, we are told $r * q = p$.
   (What would happen if you assumed it to be 'top $*$ left'?)

   (c) Below is the Cayley table for a certain group of order 6. Write down anything interesting you can tell about the group.

$*$	e	p	q	r	s	t
e	e	p	q	r	s	t
p	p	q	e	t	r	s
q	q	e	p	s	t	r
r	r	s	t	e	p	q
s	s	t	r	q	e	p
t	t	r	s	p	q	e

Please send any corrections or comments to me at glt1000@cam.ac.uk