1. Let A be a set, and let $\circ$ and $*$ be two binary operations on A, each with its own identity element. Suppose that for all $a, b, c, d \in A$, we have $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$.

   Show that $*$ and $\circ$ are in fact the same operation with the same identity element, and that this operation is associative and commutative.

2. Let X be a (non-empty) set with an associative binary operation, such that for every $x \in X$ there is a unique x' such that $xx'x = x$. Prove that X is a group.

3. (i) Show that any infinite group has infinitely many distinct subgroups.

   (ii) Give an example of an infinite group in which every element has finite order.

   (iii) Give an example of an infinite group in which every (proper) subgroup has finite order.

4. Is there a non-cyclic group of order 55? Is there a non-cyclic group of order 77?

5. Define the *order sequence* of a finite group to be a list of the orders of its elements, written in increasing order. For example, S_3 has order sequence $(1, 2, 2, 2, 3, 3)$. If two finite groups have the same order sequence, must they be isomorphic?

6. Let G be a group, $H < G$, and $g \in G$. If $gHg^{-1} \subset H$, must we have $gHg^{-1} = H$?

7. Let G be a finite group, and H a subgroup of G. Show that we can choose common representatives of the left and right cosets – that is, choose $g_1, \ldots, g_k$ such that the left cosets are $g_1H, \ldots, g_kH$ and the right cosets are $Hg_1, \ldots, Hg_k$.

8. Let G be a finite non-abelian group. Show that at most 5/8 of the pairs of elements of G commute, and show that this bound cannot be improved.

9. In how many distinct ways can the faces of a cube be coloured using at most three colours? What about a dodecahedron? (We regard as equivalent two colourings that can obtained from each other by a rotation.)

10. A *frieze group* is a group of isometries of $\mathbb{C}$ such that the real axis is invariant, and whose translation subgroup is infinite cyclic. Show that any frieze group is conjugate to one whose translation subgroup is $T = \{z \mapsto z + n : n \in \mathbb{Z}\}$.

    Show that, up to conjugation by isometries, there are exactly seven frieze groups with translation subgroup T.

11. Which finite groups have exactly two conjugacy classes? Which have exactly three? Is there an infinite group with exactly two conjuacy classes?

12. Do there exist two 2×2 matrices A and B which generate a free group? (This means that no expression of the form $A^{m_1}B^{n_1}A^{m_2}B^{n_2}\ldots A^{m_k}B^{n_k}$ equals I, for m_i, n_i non-zero integers.)

13. For which natural numbers n is there a unique group of order n?

14. Let G and H be groups such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. Show that if G and H are abelian then they must be isomorphic. If they are non-abelian, must they be isomorphic?

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