Groups : associativity

One of the rules for a group $(G, \star)$ is that the operation is *associative*. That is, for all $a, b, c \in G$, we have

$$a \star (b \star c) = (a \star b) \star c.$$

This allows us to write $a \star b \star c$ without any ambiguity.

Since we have to be careful enough to specify this as a rule for our group, it would be unwise of us just to assume that we can keep going. Perhaps there is some ambiguity in the expression $a \star b \star c \star d$?

Our operation $\star$ is applied to two things at once, so $a \star b \star c \star d$ has the following possible meanings:

1. $a \star (b \star (c \star d))$
2. $a \star ((b \star c) \star d)$
3. $(a \star b) \star (c \star d)$
4. $(a \star (b \star c)) \star d$
5. $((a \star b) \star c) \star d$

Are all five equal? It isn't immediately obvious.

Imagine you are young, and have been asked to work out, by long multiplication, the products 7×16807, 49×2401, and 343×343. Magically, all those rows of numbers you add up give the same answer! That might have surprised you all those years ago. Of course, we know now that they're all just equal to 7^6, and we *know* that multiplication of numbers works like that.

But what if they're matrices? (Until we think of suffixes, our proof that three 2×2 matrices associate tends to take half a page and use 12 variable names – so surely we can't dismiss four as obvious?) What if they're functions being composed with each other? Given that an operation as simple as subtraction fails ordinary associativity, isn't it possible that a slightly more complicated operation could be associative with three elements, but require brackets when there are four?

Well, the ordinary 'three elements associate' rule tells us that 1 and 2 are the same, and that 4 and 5 are the same. We can join them via 3, as follows.

Let $x = c \star d$. Then 3 becomes $(a \star b) \star x$, which by associativity equals $a \star (b \star x)$, which is 1.

Let $y = a \star b$. Then 3 becomes $y \star (c \star d)$, which by associativity equals $(y \star c) \star d$, which is 5.

So, to our relief, all five possibilities agree.

Now, what if we have more elements? Checking each case would become very tedious. We could perform an induction – so feel free to do so!

Please send any corrections or comments to me at glt1000@cam.ac.uk