Paper 3, Section I

1D Groups

(a) Let G be the group of symmetries of the cube, and consider the action of G on the set of edges of the cube. Determine the stabilizer of an edge and its orbit. Hence compute the order of G.

(b) The symmetric group S_n acts on the set $X = \{1, \ldots, n\}$, and hence acts on $X \times X$ by $g(x, y) = (gx, gy)$. Determine the orbits of S_n on $X \times X$.

Paper 3, Section I

2D Groups

State and prove Lagrange's Theorem.

Show that the dihedral group of order $2n$ has a subgroup of order k for every k dividing $2n$.

Paper 3, Section II

5D Groups

(a) Let G be a finite group, and let $g \in G$. Define the *order* of g and show it is finite. Show that if g is conjugate to h, then g and h have the same order.

(b) Show that every $g \in S_n$ can be written as a product of disjoint cycles. For $g \in S_n$, describe the order of g in terms of the cycle decomposition of g.

(c) Define the alternating group A_n. What is the condition on the cycle decomposition of $g \in S_n$ that characterises when $g \in A_n$?

(d) Show that, for every n, A_{n+2} has a subgroup isomorphic to S_n.

Paper 3, Section II
6D Groups

(a) Let

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \quad a, b, c, d \in \mathbb{Z} \right\},$$

and, for a prime p, let

$$SL_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \quad a, b, c, d \in \mathbb{F}_p \right\},$$

where $\mathbb{F}_p$ consists of the elements $0, 1, \ldots, p-1$, with addition and multiplication mod p.

Show that $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{F}_p)$ are groups under matrix multiplication.

[*You may assume that matrix multiplication is associative, and that the determinant of a product equals the product of the determinants.*]

By defining a suitable homomorphism from $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_5)$, show that

$$\left\{ \begin{pmatrix} 1+5a & 5b \\ 5c & 1+5d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, b, c, d \in \mathbb{Z} \right\}$$

is a normal subgroup of $SL_2(\mathbb{Z})$.

(b) Define the group $GL_2(\mathbb{F}_5)$, and show that it has order 480. By defining a suitable homomorphism from $GL_2(\mathbb{F}_5)$ to another group, which should be specified, show that the order of $SL_2(\mathbb{F}_5)$ is 120.

Find a subgroup of $GL_2(\mathbb{F}_5)$ of index 2.

Paper 3, Section II
7D Groups

(a) State the orbit–stabilizer theorem.

Let a group G act on itself by conjugation. Define the centre $Z(G)$ of G, and show that $Z(G)$ consists of the orbits of size 1. Show that $Z(G)$ is a normal subgroup of G.

(b) Now let $|G| = p^n$, where p is a prime and $n \geqslant 1$. Show that if G acts on a set X, and Y is an orbit of this action, then either $|Y| = 1$ or p divides $|Y|$.

Show that $|Z(G)| > 1$.

By considering the set of elements of G that commute with a fixed element x not in $Z(G)$, show that $Z(G)$ cannot have order p^{n-1}.

Paper 3, Section II

8D Groups

(a) Let G be a finite group and let H be a subgroup of G. Show that if $|G| = 2|H|$ then H is normal in G.

Show that the dihedral group D_{2n} of order $2n$ has a normal subgroup different from both D_{2n} and $\{e\}$.

For each integer $k \geqslant 3$, give an example of a finite group G, and a subgroup H, such that $|G| = k|H|$ and H is not normal in G.

(b) Show that A_5 is a simple group.