Paper 3, Section I

1D Groups

Show that every orthogonal 2×2 matrix R is the product of at most two reflections in lines through the origin.

Every isometry of the Euclidean plane $\mathbb{R}^2$ can be written as the composition of an orthogonal matrix and a translation. Deduce from this that every isometry of the Euclidean plane $\mathbb{R}^2$ is a product of reflections.

Give an example of an isometry of $\mathbb{R}^2$ that is not the product of fewer than three reflections. Justify your answer.

Paper 3, Section I

2D Groups

State and prove Lagrange's theorem. Give an example to show that an integer k may divide the order of a group G without there being a subgroup of order k.

Paper 3, Section II

5D Groups

State and prove the orbit–stabilizer theorem.

Let G be the group of all symmetries of a regular octahedron, including both orientation-preserving and orientation-reversing symmetries. How many symmetries are there in the group G? Let D be the set of straight lines that join a vertex of the octahedron to the opposite vertex. How many lines are there in the set D? Identify the stabilizer in G of one of the lines in D.

Paper 3, Section II

6D Groups

Let $S(X)$ denote the group of permutations of a finite set X. Show that every permutation $\sigma \in S(X)$ can be written as a product of disjoint cycles. Explain briefly why two permutations in $S(X)$ are conjugate if and only if, when they are written as the product of disjoint cycles, they have the same number of cycles of length n for each possible value of n.

Let $\ell(\sigma)$ denote the number of disjoint cycles, including 1-cycles, required when σ is written as a product of disjoint cycles. Let τ be a transposition in $S(X)$ and σ any permutation in $S(X)$. Prove that $\ell(\tau\sigma) = \ell(\sigma) \pm 1$.

Paper 3, Section II

7D Groups

Define the *cross-ratio* $[a_0, a_1, a_2, z]$ of four points a_0, a_1, a_2, z in $\mathbb{C} \cup \{\infty\}$, with a_0, a_1, a_2 distinct.

Let a_0, a_1, a_2 be three distinct points. Show that, for every value $w \in \mathbb{C} \cup \{\infty\}$, there is a unique point $z \in \mathbb{C} \cup \{\infty\}$ with $[a_0, a_1, a_2, z] = w$. Let S be the set of points z for which the cross-ratio $[a_0, a_1, a_2, z]$ is in $\mathbb{R} \cup \{\infty\}$. Show that S is either a circle or else a straight line together with ∞.

A map $J : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ satisfies

$$[a_0, a_1, a_2, J(z)] = \overline{[a_0, a_1, a_2, z]}$$

for each value of z. Show that this gives a well-defined map J with J^2 equal to the identity.

When the three points a_0, a_1, a_2 all lie on the real line, show that J must be the conjugation map $J : z \mapsto \overline{z}$. Deduce from this that, for any three distinct points a_0, a_1, a_2, the map J depends only on the circle (or straight line) through a_0, a_1, a_2 and not on their particular values.

Paper 3, Section II

8D Groups

What does it mean to say that a subgroup K of a group G is *normal*?

Let $\phi : G \to H$ be a group homomorphism. Is the kernel of ϕ always a subgroup of G? Is it always a normal subgroup? Is the image of ϕ always a subgroup of H? Is it always a normal subgroup? Justify your answers.

Let $\mathrm{SL}(2, \mathbb{Z})$ denote the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Show that $\mathrm{SL}(2, \mathbb{Z})$ is a group under matrix multiplication. Similarly, when $\mathbb{Z}_2$ denotes the integers modulo 2, let $\mathrm{SL}(2, \mathbb{Z}_2)$ denote the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}_2$ and $ad - bc = 1$. Show that $\mathrm{SL}(2, \mathbb{Z}_2)$ is also a group under matrix multiplication.

Let $f : \mathbb{Z} \to \mathbb{Z}_2$ send each integer to its residue modulo 2. Show that

$$\phi : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}_2) \ ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}$$

is a group homomorphism. Show that the image of ϕ is isomorphic to a permutation group.