3/I/1E Groups

Define the *signature* $\epsilon(\sigma)$ of a permutation $\sigma \in S_n$, and show that the map $\epsilon : S_n \to \{-1, 1\}$ is a homomorphism.

Define the *alternating group* A_n, and prove that it is a subgroup of S_n. Is A_n a normal subgroup of S_n? Justify your answer.

3/I/2E Groups

What is the *orthogonal group* $O(n)$? What is the *special orthogonal group* $SO(n)$?

Show that every element of the special orthogonal group $SO(3)$ has an eigenvector with eigenvalue 1. Is this also true for every element of the orthogonal group $O(3)$? Justify your answer.

3/II/5E Groups

For a normal subgroup H of a group G, explain carefully how to make the set of (left) cosets of H into a group.

For a subgroup H of a group G, show that the following are equivalent:

(i) H is a normal subgroup of G;

(ii) there exist a group K and a homomorphism $\theta : G \to K$ such that H is the kernel of θ.

Let G be a finite group that has a proper subgroup H of index n (in other words, $|H| = |G|/n$). Show that if $|G| > n!$ then G cannot be simple. [Hint: Let G act on the set of left cosets of H by left multiplication.]

3/II/6E Groups

Prove that two elements of S_n are conjugate if and only if they have the same cycle type.

Describe (without proof) a necessary and sufficient condition for a permutation $\sigma \in A_n$ to have the same conjugacy class in A_n as it has in S_n.

For which $\sigma \in S_n$ is σ conjugate (in S_n) to σ^2 ?

For every $\sigma \in A_5$, show that σ is conjugate to σ^{-1} (in A_5). Exhibit a positive integer n and a $\sigma \in A_n$ such that σ is not conjugate to σ^{-1} (in A_n).

3/II/7E Groups

Show that every Möbius map may be expressed as a composition of maps of the form $z \mapsto z + a$, $z \mapsto \lambda z$ and $z \mapsto 1/z$ (where a and λ are complex numbers).

Which of the following statements are true and which are false? Justify your answers.

(i) Every Möbius map that fixes ∞ may be expressed as a composition of maps of the form $z \mapsto z + a$ and $z \mapsto \lambda z$ (where a and λ are complex numbers).

(ii) Every Möbius map that fixes 0 may be expressed as a composition of maps of the form $z \mapsto \lambda z$ and $z \mapsto 1/z$ (where λ is a complex number).

(iii) Every Möbius map may be expressed as a composition of maps of the form $z \mapsto z + a$ and $z \mapsto 1/z$ (where a is a complex number).

3/II/8E Groups

State and prove the orbit–stabilizer theorem. Deduce that if x is an element of a finite group G then the order of x divides the order of G.

Prove Cauchy's theorem, that if p is a prime dividing the order of a finite group G then G contains an element of order p.

For which positive integers n does there exist a group of order n in which every element (apart from the identity) has order 2?

Give an example of an infinite group in which every element (apart from the identity) has order 2.