

Conformal Maps

Let U be an open subset of \mathbb{C} . A map $f : U \rightarrow \mathbb{C}$ is called **conformal** if it is analytic with non-zero derivative at all points in U . Note that, by the chain rule, a composition of conformal maps is conformal. We will often want our conformal maps to be injective on U , so that we have a bijection between U and $f(U)$.

1. Möbius maps

Make sure that you know the following results from IA: that Möbius maps send circles and straight lines to circles and straight lines; and that given two triples of distinct points z_1, z_2, z_3 and w_1, w_2, w_3 , there exists a unique Möbius map f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.

- (i) Let $f(z) = \frac{az + b}{cz + d}$, with $ad - bc \neq 0$. Find $f^{-1}(z)$ and $f'(z)$. Where is f conformal?
- (ii) Let $f(z) = \frac{z + 1}{z - 1}$. What are the images of the real axis, the imaginary axis, and the unit circle? What are the images of the unit disc and the quadrant $\{x + iy : x, y > 0\}$?
- (iii) Let C_1 be the unit circle, and C_2 the circle $|z - (1 + i)| = 1$. Find a Möbius map which simultaneously sends C_1 to the real axis and C_2 to the imaginary axis.

For $i = 1, 2$, let D_i be the open disc bounded by C_i . Where does your map send the region $D_1 \cap D_2$? Where does it send $\mathbb{C} \setminus (D_1 \cup D_2)$?

2. Powers

Find the effects of the following maps, using the principal branch of $z^{1/3}$ in (iii).

- (i) $f(z) = z^2$ on $\{x + iy : x, y > 0\}$
- (ii) $f(z) = z^2$ on $\{z : |z| < 1, \operatorname{Re}(z) > 0\}$.
- (iii) $f(z) = z^{1/3}$ on $\mathbb{C} \setminus \{x + iy : x \leq 0, y = 0\}$

3. exp and log

Find the effects of the following maps, using the principal branch of log in (iii) and (iv).

- (i) $f(z) = \exp(z)$ on $\{x + iy : 0 < y < \pi\}$
- (ii) $f(z) = \exp(z)$ on $\{x + iy : x > 0, 0 < y < \pi/2\}$
- (iii) $f(z) = \log(z)$ on $\{x + iy : x, y > 0\}$
- (iv) $f(z) = \log(z)$ on $\{z : |z| < 1, \operatorname{Re}(z) > 0\}$.

4. For each of the following regions U , construct a map f which sends U conformally and bijectively to the unit disc. If you construct f as a composition of several functions, it will be helpful (to both you and me) if you sketch the effect of each step.

In each case, for the points on the boundary of U where f is analytic but $f'(z) = 0$, what happens to the angles at those points?

- (i) U is the positive quadrant, $\{x + iy : x, y > 0\}$
- (ii) U is the upper half-disc, $\{z : |z| < 1, \operatorname{Im}(z) > 0\}$
- (iii) U is the open region bounded between the circles $|z - 1| = 1$ and $|z - 2| = 2$
- (iv) U is the half-strip $\{x + iy : -1 < x < 1, y > 0\}$.

(You will need only maps of the forms described in **1–3** above.)