Paper 1, Section I

2A Complex Analysis or Complex Methods

Derive the Cauchy-Riemann equations satisfied by the real and imaginary parts of a complex analytic function $f(z)$.

If $|f(z)|$ is constant on $|z| < 1$, prove that $f(z)$ is constant on $|z| < 1$.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(i) Let $-1 < \alpha < 0$ and let

$$\begin{aligned} f(z) &= \frac{\log(z-\alpha)}{z} \text{ where } -\pi \leqslant \arg(z-\alpha) < \pi\,, \\ g(z) &= \frac{\log z}{z} \quad\;\;\; \text{ where } -\pi \leqslant \arg(z) < \pi\,. \end{aligned}$$

Here the logarithms take their principal values. Give a sketch to indicate the positions of the branch cuts implied by the definitions of $f(z)$ and $g(z)$.

(ii) Let $h(z) = f(z) - g(z)$. Explain why $h(z)$ is analytic in the annulus $1 \leqslant |z| \leqslant R$ for any $R > 1$. Obtain the first three terms of the Laurent expansion for $h(z)$ around $z = 0$ in this annulus and hence evaluate

$$\oint_{|z|=2} h(z)dz\,.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(i) Let C be an anticlockwise contour defined by a square with vertices at $z = x + iy$ where

$$|x| = |y| = \left(2N + \frac{1}{2}\right)\pi,$$

for large integer N. Let

$$I = \oint_C \frac{\pi \cot z}{(z + \pi a)^4} dz.$$

Assuming that $I \to 0$ as $N \to \infty$, prove that, if a is not an integer, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^4} = \frac{\pi^4}{3\sin^2(\pi a)}\left(\frac{3}{\sin^2(\pi a)} - 2\right).$$

(ii) Deduce the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^4}.$$

(iii) Briefly justify the assumption that $I \to 0$ as $N \to \infty$.

[*Hint: For part (iii) it is sufficient to consider, at most, one vertical side of the square and one horizontal side and to use a symmetry argument for the remaining sides.*]

Paper 3, Section I

4D Complex Methods

Write down the function $\psi(u,v)$ that satisfies

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}=0\,, \qquad \psi(-\tfrac{1}{2},v)=-1, \quad \psi(\tfrac{1}{2},v)=1\,.$$

The circular arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ in the complex z-plane are defined by

$$|z+1|=1,\ z\neq 0 \ \text{ and } \ |z-1|=1,\ z\neq 0\,,$$

respectively. You may assume without proof that the mapping from the complex z-plane to the complex ζ-plane defined by

$$\zeta=\frac{1}{z}$$

takes $\mathcal{C}_1$ to the line $u=-\frac{1}{2}$ and $\mathcal{C}_2$ to the line $u=\frac{1}{2}$, where $\zeta=u+iv$, and that the region $\mathcal{D}$ in the z-plane exterior to both the circles $|z+1|=1$ and $|z-1|=1$ maps to the region in the ζ-plane given by $-\frac{1}{2}<u<\frac{1}{2}$.

Use the above mapping to solve the problem

$$\nabla^2\phi=0 \quad \text{in } \mathcal{D}, \qquad \phi=-1 \text{ on } \mathcal{C}_1 \text{ and } \phi=1 \text{ on } \mathcal{C}_2.$$

Paper 4, Section II

14D Complex Methods

State and prove the convolution theorem for Laplace transforms.

Use Laplace transforms to solve

$$2f'(t)-\int_0^t (t-\tau)^2 f(\tau)\,d\tau = 4tH(t)$$

with $f(0)=0$, where $H(t)$ is the Heaviside function. You may assume that the Laplace transform, $\widehat{f}(s)$, of $f(t)$ exists for $\operatorname{Re} s$ sufficiently large.

Paper 4, Section I

4E Complex Analysis

Let $f(z)$ be an analytic function in an open subset U of the complex plane. Prove that f has derivatives of all orders at any point z in U. [You may assume Cauchy's integral formula provided it is clearly stated.]

Paper 3, Section II

13E Complex Analysis

Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous function such that

$$\int_{\Gamma} g(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a rectangle in $\mathbb{C}$ with sides parallel to the real and imaginary axes. Prove that g is analytic.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous. Suppose in addition that f is analytic at every point $z \in \mathbb{C}$ with non-zero imaginary part. Show that f is analytic at every point in $\mathbb{C}$.

Let $\mathbb{H}$ be the upper half-plane of complex numbers z with positive imaginary part $\Im(z) > 0$. Consider a continuous function $F : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ such that F is analytic on $\mathbb{H}$ and $F(\mathbb{R}) \subset \mathbb{R}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} F(z) & \text{if } \Im(z) \geqslant 0 \\ \overline{F(\overline{z})} & \text{if } \Im(z) \leqslant 0. \end{cases}$$

Show that f is analytic.